



DEPARTMENT OF EDUCATION

GRADE 11 ADVANCE MATHEMATICS

11.4: GEOMETRY



FODE DISTANCE LEARNING



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GRADE 11

ADVANCE MATHEMATICS

MODULE 4

GEOMETRY

TOPIC 1: BASIC GEOMETRY CONCEPTS

TOPIC 2: CONGRUENCY AND SIMILARITY

TOPIC 3: CIRCLES

TOPIC 4: SOLID GEOMETRY



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DIANA TEIT AKIS

PRINCIPAL



Flexible Open and Distance Education
Papua New Guinea

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SECRETARY'S MESSAGE

Achieving a better future by individual students and their families, communities or the nation as a whole, depends on the kind of curriculum and the way it is delivered.

This course is a part of the new Flexible, Open and Distance Education curriculum. The learning outcomes are student-centred and allows for them to be demonstrated and assessed.

It maintains the rationale, goals, aims and principles of the national curriculum and identifies the knowledge, skills, attitudes and values that students should achieve.

This is a provision by Flexible, Open and Distance Education as an alternative pathway of formal education.

The course promotes Papua New Guinea values and beliefs which are found in our Constitution, Government Policies and Reports. It is developed in line with the National Education Plan (2005 -2014) and addresses an increase in the number of school leavers affected by the lack of access into secondary and higher educational institutions.

Flexible, Open and Distance Education curriculum is guided by the Department of Education's Mission which is fivefold:

- To facilitate and promote the integral development of every individual
- To develop and encourage an education system satisfies the requirements of Papua New Guinea and its people
- To establish, preserve and improve standards of education throughout Papua New Guinea
- To make the benefits of such education available as widely as possible to all of the people
- To make the education accessible to the poor and physically, mentally and socially handicapped as well as to those who are educationally disadvantaged.

The college is enhanced to provide alternative and comparable pathways for students and adults to complete their education through a one system, many pathways and same outcomes.

It is our vision that Papua New Guineans' harness all appropriate and affordable technologies to pursue this program.

I commend all those teachers, curriculum writers, university lecturers and many others who have contributed in developing this course.

UKE KOMBRA, PhD

Secretary for Education



UNIT INTRODUCTION

Geometry is a branch of Mathematics that has a lot of applications in our everyday lives such as measuring the size or shape of anything. We use Geometry in building our house, bridges, roads, and different means of transportation.

The word geometry comes from the Greek words *geo* which means “earth” and *metron* which means “to measure”. It is the study of shapes and figures as well as their properties. In this sense, geometry plays a vital role in making our world a better place to live in.

The study of geometry will show us how to develop ideas through logical reasoning and critical thinking than the conventional methods like description and observation.

To study geometry, you have to know familiar and unfamiliar terms, as well as the theorems and postulates that are building blocks of geometry but if you follow the step-by-step procedures in this module and complete all the activities, learning geometry can be fun and challenging.

The topics include:

TOPIC 1: BASIC GEOMETRY CONCEPTS

In this topic we will discuss undefined terms such as points, lines and planes. Then use those terms to define and describe angles, triangles, quadrilaterals and polygons. The axioms, postulates and theorems will guide us through the study of geometry.

TOPIC 2: CONGRUENCY AND SIMILARITY

In this topic we will define congruency and similarity. Then use geometric properties and methods to show proof of congruency and similarity of triangles, which you can then apply to prove congruency and similarity of other planes. Theorems associated with similarity and congruence, and techniques such as k – factor will aid us in solving real life problems.

TOPICS 3: CIRCLES

In this topic we will define a circle and investigate its association with other planes. Certain theorems will be discussed in relation to inscribe circle, circumscribe circle, tangents and secants will provide us the basis for navigational purpose and geographical study. Having a good foundational knowledge in circles will ease understanding of ellipse and planet movements and weather patterns.

TOPICS 4: SOLID GEOMETRY

In this topic we will discuss 2-D and 3- D shapes. We will discuss lines and planes and their importance in 2-D and 3-D shapes. Knowing features and knowledge of specific properties of certain solids will provide us the prerequisite to apply basic three trigonometric ratios to solve mathematical, industrial and economic, or social problems we will encounter.

Geometry is one of the oldest topics studied in mathematics.



Student Learning Outcomes

On successful completion of this module, students will be able to:

- identify and describe the undefined terms, and other geometric figures,
- draw, illustrate and name points, lines and planes and other geometric figures,
- recognize collinear and coplanar points,
- classify angles and triangles according to sides and according to angles,
- compute the sum of interior and exterior angles of a polygon.
- identify similarity and congruency among triangles,
- discuss the properties of quadrilaterals and other geometric figures,
- develop knowledge and understanding of triangle congruence and similarities, parallelism, quadrilaterals, circles, solid figures and the like, and
- find the area, circumference, arc length and other parts of a circle.



Time Frame

This unit should be completed within 10 weeks.

If you set an average of 3 hours per day, you should be able to complete the unit comfortably by the end of the assigned week.

Try to do all the learning activities and compare your answers with the ones provided at the end of the unit. If you do not get a particular exercise right in the first attempt, you should not get discouraged but instead, go back and attempt it again. If you still do not get it right after several attempts then you should seek help from your friend or even your tutor. Do not pass any question without solving it first.



11.4. 1: BASIC GEOMETRIC CONCEPTS

Undefined terms are basic terms in geometry which cannot be defined but can be described. Some undefined terms are points, lines and planes. These undefined terms form the basis of our study of the subject and for defining other geometric terms.

11.4.1.1 Points, Lines and Planes

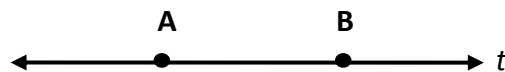
A **point** is the simplest figure we study in geometry. It represents an exact location in space. It could be represented by a dot, the tip of a newly sharpened pencil, a particle of dust, or the full stop at the end of a sentence. Thus, a point has no dimension for it has no length, no **size**, no width, no height nor thickness.

A point is named using a capital letter.

Example:

- **A** is read as “point **A**”
- **B** is read as “point **B**”

Suppose you connect points **A** and **B** with a line. This line could be extended in both directions as shown in the figure.



A **line** is a connection of collinear points that extends indefinitely in either direction. It is drawn with arrowheads to indicate that it is continuously going in opposite directions. A line can be represented by the edge of a table, a stretched piece of thread or a strand of hair. It can be named using any two points on the line that are collinear. **Collinear** means lying on one line. Thus, the figure above is named line $\overline{AB}$ or line $\overline{BA}$ and can be written in symbol as **AB** or **BA**. Note that the given figure of a line can also be named using a single lower case letter such as line t .

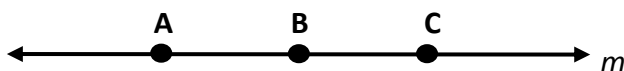
Remember that when we talk of a line in this lesson, we always refer to a **straight line**.

The terms *point* and *line* were described above but were not defined. Important terms in geometry are defined.

A **definition** is a statement, description or meaning of a term stated clearly and unambiguously, so that all those using it will clearly understand it. It is important to understand the meaning of a term or a concept that forms the basis in order to draw understanding in all the other concepts or properties that contains the term.



Refer to the figure below.



This line can be described as line AC or line CA , line AB or line BA , line BC or line CB , or line m . Although there are three given points on the line, this line has infinite number of points. The existence of the given points on the line can partition the line into subsets, namely segment and ray.

Definition:

A **segment** (or **line segment**) is a part of the line that has two endpoints.

In the figure above, we have:

$\overline{AB} = \overline{BA}$ (read as segment **AB** is the same as segment **BA**)

$\overline{AC} = \overline{CA}$ (read as segment **AC** is the same as segment **CA**)

$\overline{BC} = \overline{CB}$ (read as segment **BC** is the same as segment **CB**)

Symbolically, it is important to draw a small segment (bar) above the two letters in naming a segment. Since the line segment has two endpoints, it can be named starting from any of the endpoints. Likewise, the endpoints indicate that the segment is measurable and is finite. Thus, segment **AB** refers to all points from point **A** to point **B** only.

Definition:

A **ray** is a part of a line that has one endpoint and extends infinitely in one direction.

In the given figure above, we have:

$\overrightarrow{AB}$ (read as ray **AB**)

$\overrightarrow{BA}$ (read as ray **BA**)

$\overrightarrow{AC}$ (read as ray **AC**)

$\overrightarrow{CA}$ (read as ray **CA**)

$\overrightarrow{BC}$ (read as ray **BC**)

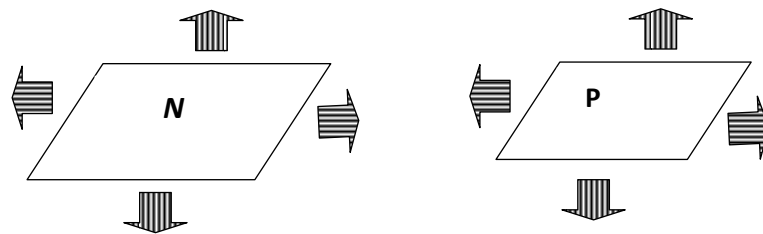
$\overrightarrow{CB}$ (read as ray **CB**)

In naming a ray, it is important to draw a small ray above the two letters pointing to the right direction. Although talking about segments, $\overline{AB} = \overline{BA}$, $\overrightarrow{AB} \neq \overrightarrow{BA}$ because the emphasis in naming a ray starts from its endpoint. Thus, the endpoint of ray **AB** is point **A** while the endpoint of ray **BA** is point **B**. However, $\overrightarrow{AB} = \overrightarrow{AC}$ and $\overrightarrow{CB} = \overrightarrow{CA}$ since the emphasis is on the endpoint. The opposite rays **BA** and **BC** have the common endpoint but we cannot say that ray **BA** = **BC** because they are in opposite direction.



If two points are joined by a straight line, the line therefore lies on a flat surface called **plane**. A plane can be represented by the top of a table, the surface of a wall, or a ceiling. These surfaces show that a plane is flat but they do not show that a plane extends indefinitely.

Thus, a plane, just like the point and line, is called an undefined term because it is not possible to draw a picture of a plane that extends indefinitely. The given figures will show how a plane is named. The arrows in the figure indicate that a plane can be extended endlessly in all directions.



A plane may be named by a capital letter like plane P (but not necessarily a point on the plane) or by any three given **non-collinear** but **coplanar** points on the plane as plane **ABC**.

Coplanar points are points that lie on the same plane.

Points, lines, and planes are the undefined terms used in **plane geometry** which is the study of properties and relationships of points, lines and planes and of figures that can be represented on a plane.

Can you name things you see around that constitute points, lines and planes?

Now that you have an idea about how **undefined** terms are described and explained, any **new term** that we will subsequently use will be defined.



LEARNING ACTIVITY 11.4.1.1



5 minutes

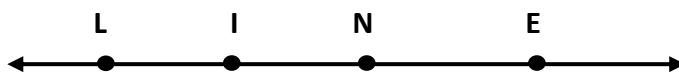
1. Write whether each of the following best represents a *point*, a *line* or a *plane*.

- | | |
|-----------------------|-------|
| a. tip of a needle | _____ |
| b. edge of a table | _____ |
| c. electric wire | _____ |
| d. a window pane | _____ |
| e. the wall of a room | _____ |



- f. strand of hair _____
- g. sewing thread _____
- h. grains of sand _____
- i. mole on your face _____
- j. a road map _____

2. Refer to the figure below and **write** whether the following statements are **true** or **false**.



- a. $\overline{IN}$ is the same as $\overline{NI}$. _____
- b. $\overleftrightarrow{LI}$ is the same as $\overleftrightarrow{NE}$. _____
- c. $\overrightarrow{IN}$ is the same as $\overrightarrow{NE}$. _____
- d. $\overrightarrow{IL}$ and $\overrightarrow{NE}$ are opposite rays. _____
- e. $\overrightarrow{IE}$ and $\overrightarrow{EI}$ are the same. _____

3. Draw and label the following :

- a. line **t**
- b. point **N**
- c. plane **R**
- d. point **Z** lying on a line **m**
- e. a line **PT** lying on plane **Q**



11.4.1.2 Naming Angles and Triangles

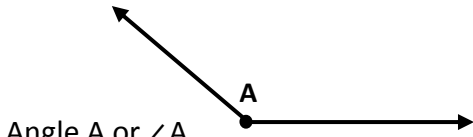
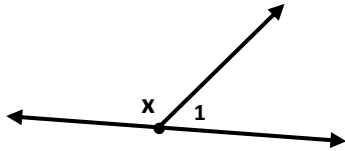
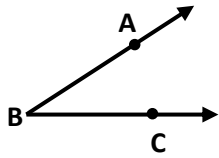
One of the most common applications of geometry is in Navigation, because navigators make use of angles to calculate position and determine the navigational path they will take.

Let us make use of the line and its subsets to determine plane figures like angles and triangles.

Definition:

An **angle** is a geometric figure formed by two rays that have a common endpoint. The rays are the **sides** of the angle and the common endpoint is its **vertex**. It is denoted by the symbol $\angle$.

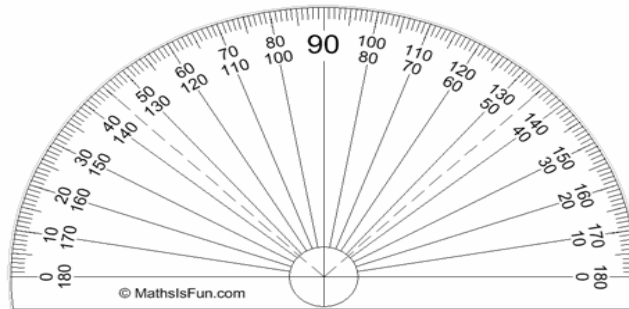
The two rays that form an angle may be thought of as one ray that moves from a starting position about the vertex. The ray in the starting position is called the **initial side** while the second ray is called the **terminal side**.

WAYS OF NAMING ANGLES	ILLUSTRATION
1. by using a capital letter at the vertex	 Angle A or $\angle A$
2. by using a small letter or a numeral	 Angle x and Angle 1
3. by using the three letters associated with the sides and the vertex	 Angle ABC or $\angle ABC$



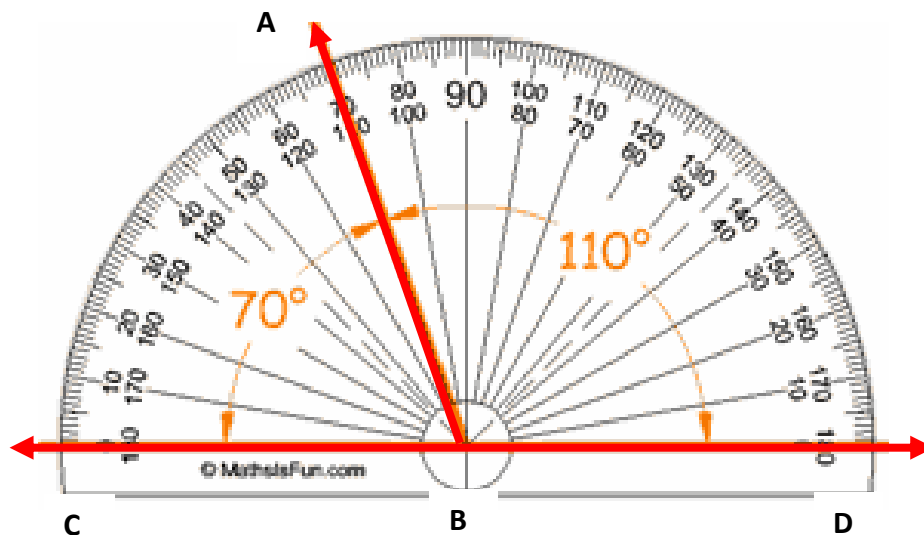
Arcs at angles

We can use a **protractor** to find the measure of an angle in **degrees**. Remember that the measure of an angle is a real number between 0 and 180 degrees. Many protractors have two scales, one reading from left to right, and the other reading from right to left.



To measure an angle with a protractor, follow these steps:

1. Align the protractor carefully with one side of the angle.
2. **Ensure** the center of the protractor is on the vertex of the angle.
3. Read the measure on the correct scale. Use the scale that has its zero point on the side of the angle aligned with the protractor.



The figure above shows that $m\angle ABC = 70$ and $m\angle ABD = 110$. In writing the measure of an angle, you need to write the lower case letter m then, remove the unit of degree ($^\circ$).

In case you need to write the degree, it should be written as $\angle ABC = 70^\circ$ and $\angle ABD = 110^\circ$.

Remember:

When you measure angles, you are measuring the number of degrees in the angle, not the length of the sides. The length of the sides does not matter.



Look at these two angles. Which is larger Angle **A** or angle **B**?

Angle **B** is larger than angle **A**. Angle **B** is larger than angle **A**. Angle

A has longer sides, but angle **B** contains more degrees



If two angles say $\angle A$ and $\angle B$ have equal measure, then they are said to be **congruent** to each other. In symbol, we write $\angle A \cong \angle B$ (read as “angle **A** is congruent to angle **B**”). If both angles measure 80 degrees (or 80°), we write $m\angle A = m\angle B$. Note that we use the congruent ($\cong$) symbol if no measurement is involved and the equal ($=$) sign if it involves measurement. Thus, if $\angle A \cong \angle B$, then $m\angle A = m\angle B$.

Angles are classified according to their measures.

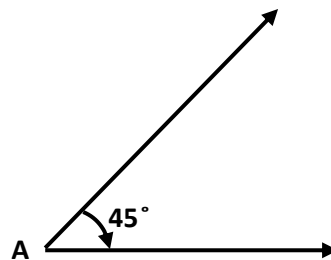
Acute Angle – an angle with a measure less than 90° .

Right Angle – an angle with a measure of exactly 90° .

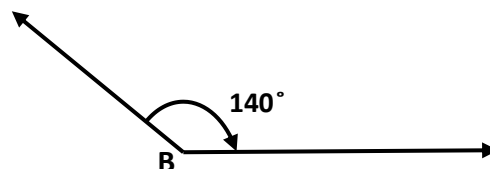
Obtuse Angle – an angle with a measure between 90° and 180° .

Examples

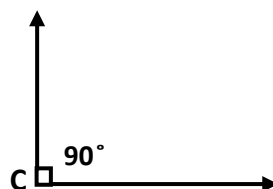
1. $\angle A$ is an acute angle.



2. $\angle B$ is an obtuse angle.



3. $\angle C$ is a right angle.

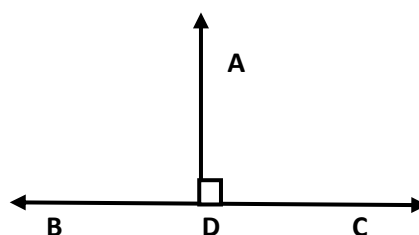
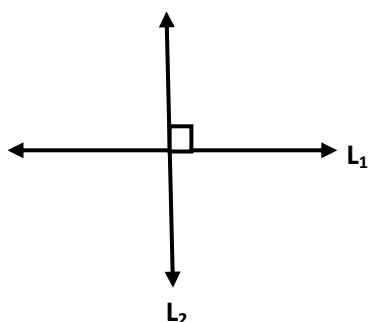




To restate the definition above, we can say that $m\angle A < 90$, $m\angle B > 90$ and $m\angle C = 90$. Since $\angle C$ is a right angle, then the two arms of the angle are perpendicular to each other.

If two intersecting lines, rays or segments form at least one right angle, then they are **perpendicular** to each other.

The following figures illustrate the concept of perpendicularity:



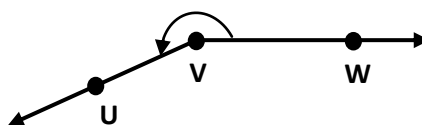
$L_1 \perp L_2$ and $\overrightarrow{DA} \perp \overrightarrow{BC}$. Perpendicularity can also be determined between a line and a segment, a line and a ray, and a ray and a segment.

Aside from acute, obtuse and right angles, there are also special angles that we have yet to define. We have a **straight angle** whose measure is 180° and forms a straight line.

On the other hand, an angle whose measure is between 180° and 360° is called **reflex angle**. In a reflex angle, the rotation from the initial side to the terminal side is indicated so that we know that the angle is more than 180° .



$\angle RST$ is a straight angle.



$\angle WVU$ is a reflex angle.

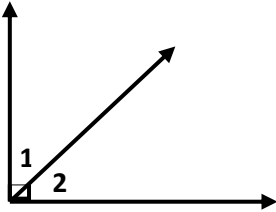
Knowing the kinds of angles, let us now study how angles relate to each other. In arithmetic, we have learnt that a number can be expressed as a sum of two numbers like $90 = 30 + 60$.

If we express the addends as degree measures, we know that 30° and 60° are both acute angles and the sum of 90° is a right angle. So we can say that a right angle can be separated into two acute angles taken as an angle pair.

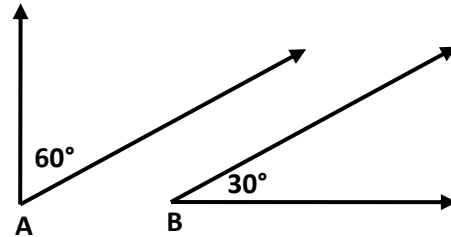
Certain angle pairs are given special names based on their relative position to one another or based on the sum of their respective measures.



Two angles are **complementary** if the sum of their degree measures is 90, and each angle is the **complement** of the other.



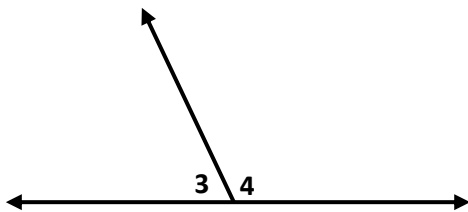
$\angle 1$ and $\angle 2$ are complementary
 $m\angle 1 + m\angle 2 = 90$



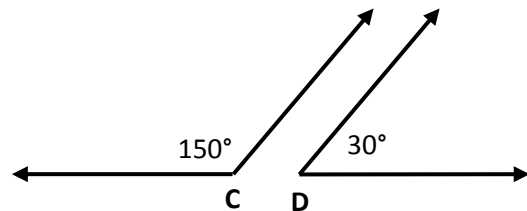
$\angle A$ and $\angle B$ are complementary
 $m\angle A + m\angle B = 90$

It is not correct to call three angles complementary even though the sum of their measures is 90°. **Complementary angles** mean two angles, each of which is called a complement of the other. This also applies to supplementary angles.

Two angles are **supplementary** if the sum of their degree measures is 180.



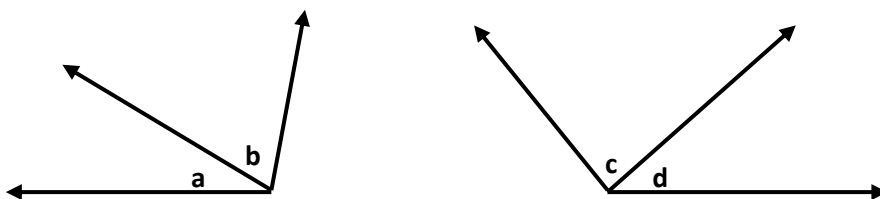
$\angle 3$ and $\angle 4$ are supplementary
 $m\angle 3 + m\angle 4 = 180$



$\angle C$ and $\angle D$ are supplementary
 $m\angle C + m\angle D = 180$

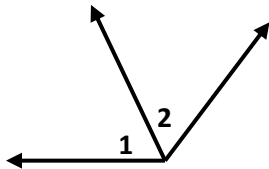
For complementary and supplementary angles, the two angles which are complements or supplements of each other may or may not have a common arm. In the next two definitions, we have angles with a common arm.

Adjacent angles are angles that share a common arm and have the same vertex but have no interior points in common.

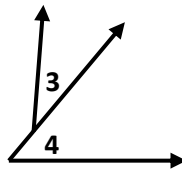




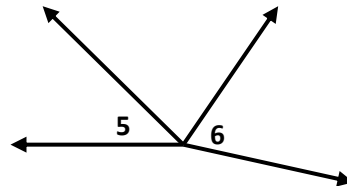
Adjacent means *next to*. But how are two angles next to each other considered adjacent? The following figures contrast three pairs of angles.



$\angle 1$ and $\angle 2$ are adjacent angles.

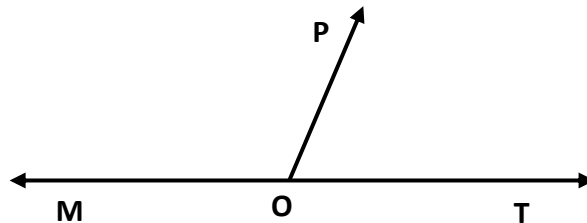


$\angle 3$ and $\angle 4$ are not adjacent because they do not share the same vertex.



$\angle 5$ and $\angle 6$ are not adjacent because they do not share a common side.

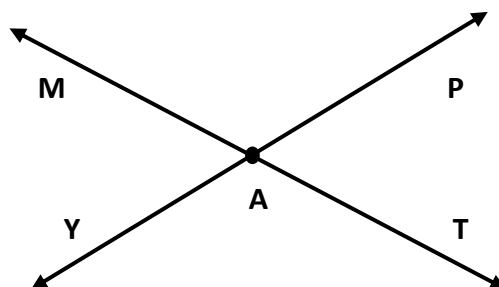
A **linear pair** of angles is formed by two adjacent angles that share a common side and their other arms are opposite rays forming a straight angle.



$\angle MOP$ and $\angle POT$ form a **linear pair**. Likewise, they are **supplementary** and **adjacent** angles.

The last relationship of angles that we will tackle is the case of vertically opposite angles.

Vertically opposite angles are two non-adjacent angles formed by two intersecting lines. Likewise, two angles are vertically opposite angles if the sides of one angle are the opposite rays of the sides of the other angle.





The vertically opposite angles in the figure are $\angle MAY$ and $\angle PAT$, as well as $\angle MAP$ and $\angle YAT$. While angles in a linear pair are supplementary, **vertically opposite angles are congruent**. Since these pairs of angles are vertical opposite angles, therefore $\angle MAY \cong \angle PAT$ and $\angle MAP \cong \angle YAT$. Thus, we can also say that $m\angle MAY = m\angle PAT$ and $m\angle MAP \cong m\angle YAT$.

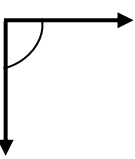
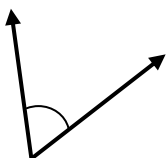
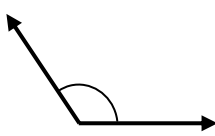
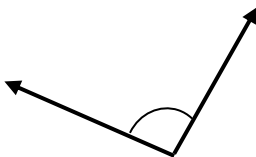
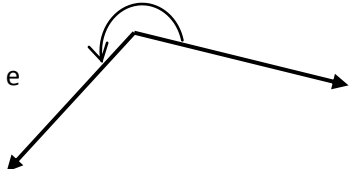
**LEARNING ACTIVITY 11.4.1.2**

10 minutes

1. Identify the kinds of angles according to measure.

- a. 39° _____
b. 124° _____
c. 90° _____
d.  _____
e.  _____

2. Measure the following angles and indicate the measurement on the blank provided.

- a.  _____
- c.  _____
- b.  _____
- d.  _____
- e.  _____



3. Draw the following angles with their corresponding measure.

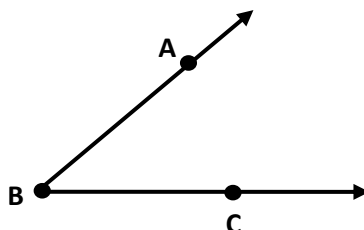
- a. One acute angle
- b. Two right angles
- c. An angle whose measure is 30° less than 100° .
- d. An angle whose measure is 125° .
- e. Two congruent obtuse angles

4. Write whether the statement is **TRUE** or **FALSE**.

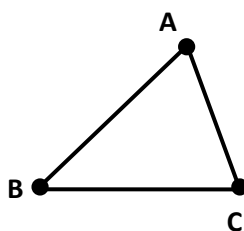
- _____ a. If two angles are acute, then they are complementary.
- _____ b. If two angles are complementary, then they are congruent.
- _____ c. If two angles are supplementary, then they form a linear pair.
- _____ d. The supplement of an obtuse angle is also obtuse.
- _____ e. If two vertical angles are supplementary, then they are both right angles.



Revisiting the definition of an angle, in the figure below, $\angle ABC$ is formed by two rays (ray BA and ray BC) which form the sides, and with the vertex B.



If we connect the points A and C, cut off the rays and form a figure composed of three segments, the figure we form is shown below.



Description of vertex

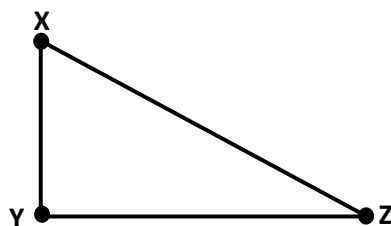
In this case, angles could be formed by segments and not rays as their arms. The figure we formed is a triangle with **vertices A, B, and C**, and the sides are $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$. There are three angles formed by the segments and named as $\angle ABC$, $\angle BCA$ and $\angle CAB$. The triangle itself could be denoted as $\triangle ABC$ (or $\triangle CBA$), or $\triangle BCA$ (or $\triangle ACB$), or $\triangle CAB$ (or $\triangle BAC$). The common form of triangle identification is by describing the triangle in alphabetical order as $\triangle ABC$.

A **triangle** is a polygon with three sides.

A **polygon** is a simple closed plane figure made up of line segments and vertices.

Example

In the given figure below, name the triangle in different ways. Name also the sides, vertices and angles of the triangle.





Solution

The triangle can be named as $\triangle XYZ$ (or $\triangle ZYX$), or $\triangle YZX$ (or $\triangle XZY$), or $\triangle ZXY$, (or $\triangle YXZ$).

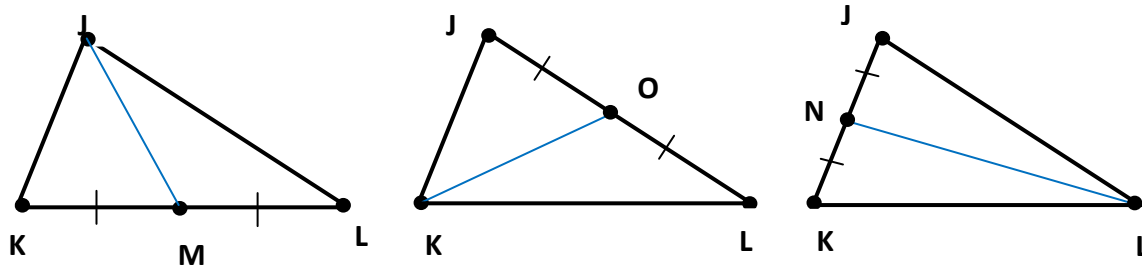
The sides are $\overline{XY}$, $\overline{YZ}$ and $\overline{XZ}$.

The vertices are points **X**, **Y**, and **Z**.

The angles are $\angle XYZ$, $\angle YZX$ and $\angle ZXY$.

Aside from vertices, angles and sides, triangles also have medians, angle bisectors and altitudes. They are defined as follows:

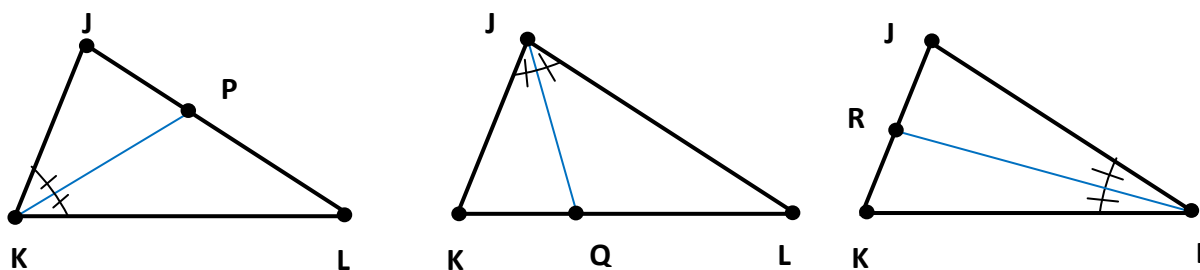
A **median** is a segment drawn from one vertex to the midpoint of the opposite side.



Looking at the figure of $\triangle JKL$, the vertices are points **J**, **K** and **L** while the midpoints are **M**, **N** and **O**. Just remember that median means “middle”. A median of a triangle is simply a segment drawn from one vertex to the “middle” (midpoint) of the opposite side.

Therefore, we can say that $\overline{JM}$, $\overline{KO}$ and $\overline{LN}$ are medians of $\triangle JKL$. Take note that matching parts of the figure with a dash through it indicate that they have the same measure and they are congruent like $\overline{KM} \cong \overline{ML}$ since $\overline{KL}$ is being bisected by midpoint **M** or median $\overline{JM}$.

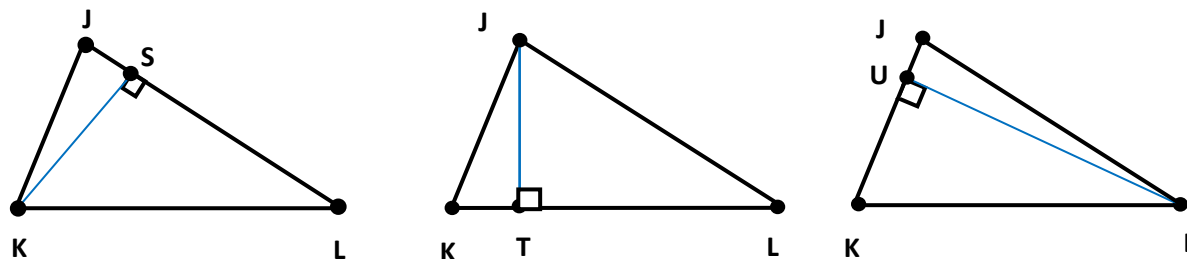
An **angle bisector** is a segment from one vertex to the opposite side that bisects an angle of a triangle.



$\overline{KP}$, $\overline{JQ}$, and $\overline{LR}$ are angle bisectors of $\triangle JKL$. To “bisect” means “to divide into equal parts”. An angle bisector divides one angle of a triangle into two equal angles.



“Altitude” is the same as “height”. To measure the altitude or height of a tower or a building, you just get the vertical distance from the ground to the top of the structure.



An **height** is a segment drawn from one vertex that is perpendicular to the opposite side.

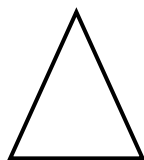
The heights of $\triangle JKL$ are $\overline{KS}$, $\overline{JT}$ and $\overline{LU}$

Types of Triangles by Angle Property

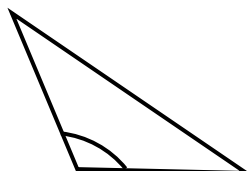
An **acute triangle** is a triangle composed of three acute angles.

An **obtuse triangle** is a triangle with one obtuse angle.

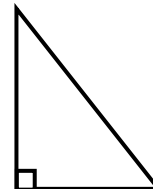
A **right triangle** is a triangle with one right angle.



Acute triangle



Obtuse triangle



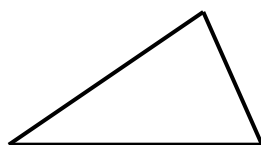
Right triangle

Types of Triangles by Side Property

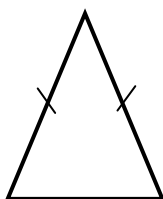
A **scalene triangle** is a triangle with no two equal sides.

An **isosceles triangle** is a triangle with two equal sides.

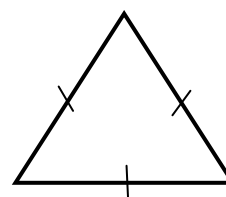
An **equilateral triangle** is a triangle with three equal sides.



Scalene triangle



Isosceles triangle

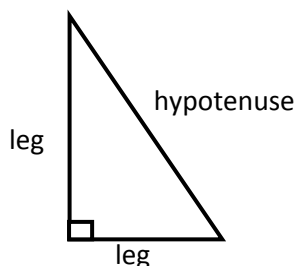


Equilateral triangle

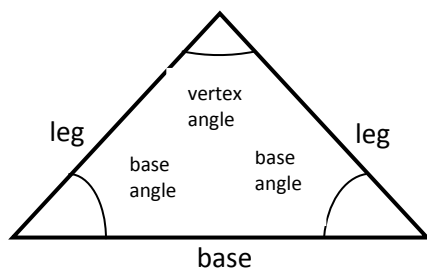


An equilateral triangle has three equal angles. Thus, it is also called **equiangular triangle**. Each triangle has three sides, three vertices and three angles. Right triangles and isosceles triangles have special names for some of their parts.

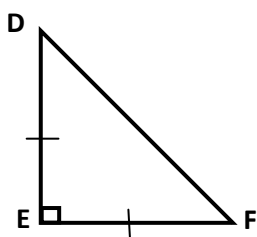
In a right triangle, the longest side, or the side opposite the right angle is called the **hypotenuse**. The other two sides are called the **legs** of the right triangle.



In an isosceles triangle, the two sides that are equal in length are called the **legs** and the third side is called the **base**. The angle formed by the two legs is called the **vertex angle** while the two other angles (which are equal in measure) are called the **base angles**.



Example The figure below is an isosceles right triangle with $\angle E$ as the right angle.



Identify:

- a) the hypotenuse
- b) the base
- c) the legs
- d) the vertex angle
- e) the base angles

Solution

- a) The hypotenuse is the side opposite the right angle. Thus, the hypotenuse is $\overline{DF}$.
- b) The base of an isosceles triangle is defined as the side opposite the vertex angle.
The hypotenuse $\overline{DF}$ is also the base of $\triangle DEF$.
- c) Since $\overline{DF}$ is both the base and hypotenuse, the legs are $\overline{DE}$ and $\overline{EF}$.



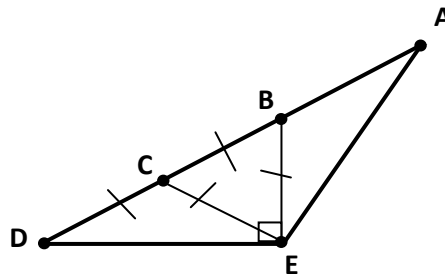
d) The vertex angle of an isosceles right triangle is the right angle which is $\angle E$. It could also be expressed as $\angle DEF$ or $\angle FED$.

e) The base angles are the two other angles $\angle D$ and $\angle F$.

**LEARNING ACTIVITY 11.4.1.2**

10 minutes

1. Refer to the given figure and answer the following:



- Identify the equilateral triangle.
- Identify the obtuse triangle.
- Is $\triangle CDE$ scalene?
- Is $\triangle BCE$ an acute triangle?
- Is $\triangle BDE$ a right triangle?
- Name an isosceles triangle.
- Which segment could be a hypotenuse?
- $\overline{EC}$ is the median of which triangle?
- How many triangles are there in the figure?
- Name all triangles in the figure.

2. Draw and label the following:

- an isosceles triangle



b. a right triangle

c. an acute isosceles triangle

d. an isosceles right triangle

e. an obtuse isosceles triangle

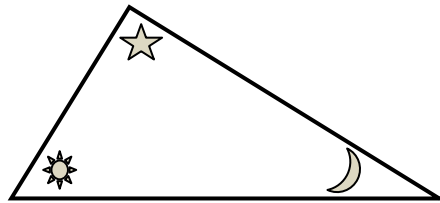


11.4.1.3 Angle Sum

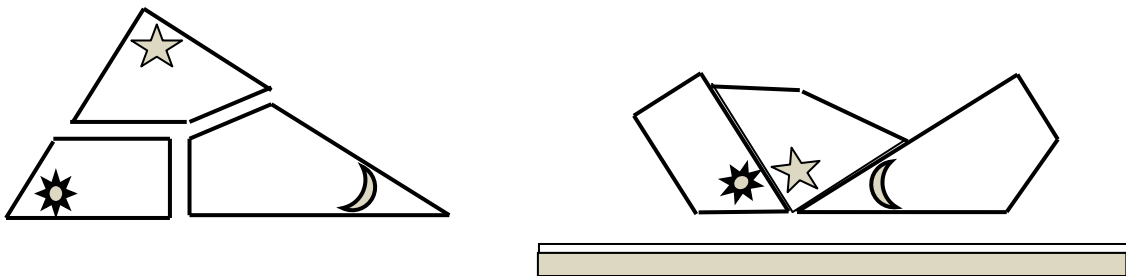
There are three angles in every triangle. Since these three angles are inside the triangle, they are called **interior angles** of a triangle. There is also a unique relationship among the measures of these three angles.

Let us do this activity and discover some important facts about triangles.

Draw a triangle on a colored paper and draw pictures at each corner so that we can identify them.



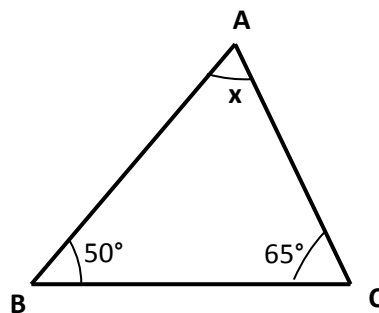
Now cut the triangles into three portions so that each portion contains only one of the vertices of the triangle. Put these three corners together against the side of a straight edge.



From the activity, we would conclude that the three angles form a straight angle. Since a straight angle is exactly 180° , we can say that the angles in a triangle give a sum of 180° .

The sum of the interior angles of a triangle is 180° .

Example: Find the $m\angle x$ in the given isosceles triangle.





Solution

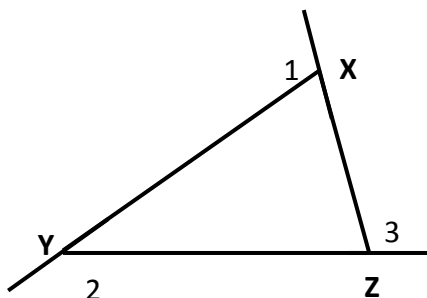
$$\text{In } \triangle ABC, \angle x + 50^\circ + 65^\circ = 180^\circ$$

$$\angle x + 115^\circ = 180^\circ$$

$$\angle x = 180^\circ - 115^\circ$$

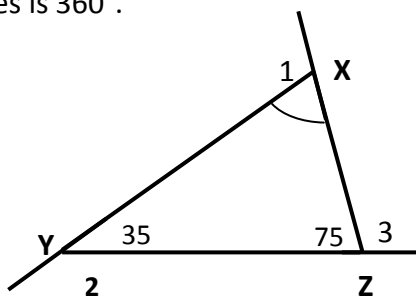
$$\angle x = 65^\circ$$

Let us distinguish between an interior angle and an exterior angle of a polygon. In $\triangle XYZ$, the interior angles are $\angle X$, $\angle Y$ and $\angle Z$. If we extend the sides of the triangle we form the exterior angles $\angle 1$, $\angle 2$ and $\angle 3$.



Whatever the shape or size of $\triangle XYZ$ is, the sum of the interior angles of a triangle is 180° . On the other hand, the sum of the exterior angles of a triangle is 360° . The interior and exterior angles are supplementary.

Example In triangle XYZ, $m\angle Y = 35$ and $m\angle Z = 75$. Find the $m\angle X$ and check if the sum of the exterior angles is 360° .



Solution

$$\text{In } \triangle XYZ, \angle X + 35^\circ + 75^\circ = 180^\circ$$

$$\angle X + 110^\circ = 180^\circ$$

$$\angle X = 180^\circ - 110^\circ$$

$$\angle X = 70^\circ$$

$$m\angle X = 70$$

Since the exterior angle of an interior angle is its supplement because they form a linear pair, therefore $\angle X$ and $\angle 1$ are supplementary, that is, if $\angle X = 70^\circ$, then $\angle 1 = 110^\circ$. Likewise, if $\angle Y = 35^\circ$, then $\angle 2 = 145^\circ$ and if $\angle Z = 75^\circ$, then $\angle 3 = 105^\circ$.



To check if the sum of the exterior angles is 360° , we have

$$\begin{aligned}\angle 1 + \angle 2 + \angle 3 &= 360^\circ \\ 110^\circ + 145^\circ + 105^\circ &= 360^\circ \\ 360^\circ &= 360^\circ\end{aligned}$$

You have learnt that triangles have three exterior angles. Each exterior angle equals the sum of opposite interior angles. The **opposite interior angles** refer to the two angles of a triangle that do not form a linear pair with respect to the given exterior angle. Using the same figure of $\triangle XYZ$, the opposite interior angles of $\angle 1$ are $\angle Y$ and $\angle Z$; the remote interior angles of $\angle 2$ are $\angle X$ and $\angle Z$; and the remote interior angles of $\angle 3$ are $\angle X$ and $\angle Y$.

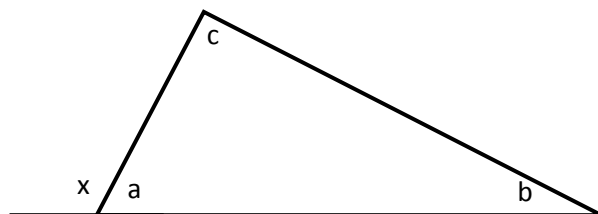
There is a special relationship between the measures of opposite interior angles and their corresponding exterior angle. That is the degree measure of an exterior angle of a triangle is equal to the sum of the degree measures of the two opposite interior angles.

Exterior angle = Sum of opposite interior angles (use $\angle x = \angle a + \angle b$, $\angle y = \angle b + \angle c$, $\angle z = \angle a + \angle c$)

$\angle 1 = \angle Y + \angle Z$	$\angle 2 = \angle X + \angle Z$	$\angle 3 = \angle X + \angle Y$
$110^\circ = 35^\circ + 75^\circ$	$145^\circ = 70^\circ + 75^\circ$	$105^\circ = 70^\circ + 35^\circ$
$110^\circ = 110^\circ$	$145^\circ = 145^\circ$	$105^\circ = 105^\circ$

Exterior and Opposite Interiors

The exterior angle of a triangle equals the sum of the two opposite interior angles.



X is exterior angle
a is the interior angle
b and c are opposite interior angles

$$\begin{aligned}X + a &= 180 & [1] \\ a + b + c &= 180 & [2]\end{aligned}$$

$$\begin{aligned}a &= 180 - x & \text{From [1]} \\ 180 - x + b + c &= 180 & \text{Substitute for a in [2]} \\ b + c &= 180 - 180 + x \\ b + c &= x\end{aligned}$$

Therefore x is equal to the sum of the two opposite interiors (b + c).

**LEARNING ACTIVITY 11.4.1.3**

10 minutes

Solve the following problems.

1. The two interior angles of a triangle have measures of 47° and 54° . Find the measure of the third angle and the measures of the exterior angles.
2. The measure of one exterior angle at the base of an isosceles triangle is 130° . Find the measure of each of the angles.
3. The measure of one acute angle of a right triangle is two times the measure of the other. Find the measure of each acute angle. (no example provided)
4. What is the measure of an exterior angle of an equilateral triangle?
5. Solve for x .



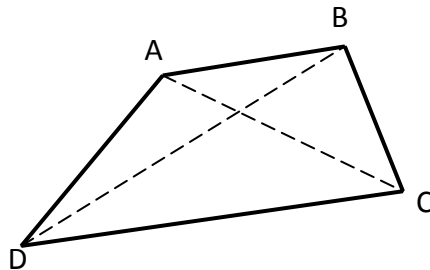
6. Given that two exterior angles of a triangle are 147° and 106° , find the third exterior angle.
7. If an exterior of a triangle is 60° and an opposite interior is 45° , find the other opposite interior angle.



11.4.1.4 Quadrilaterals

You have learnt previously about triangles. If a triangle is a three-sided polygon, this time, you will learn about geometric figures with four sides. Quadrilaterals, like triangles, are polygons that are useful in our life. Designs on different parts of buildings and furniture at home are rectangular or square in shape. Let us now study the different kinds of quadrilaterals and their properties.

A **quadrilateral** is a polygon with four sides, four angles and four vertices. The sides that are not adjacent are called **opposite sides**. Angles having a common side are called **adjacent angles**.



In the figure above, a quadrilateral uses its four consecutive vertices as quadrilateral ABCD. A segment connecting two opposite vertices indicated by the dashes are diagonals of the quadrilateral.

In quadrilateral ABCD, the two pairs of opposite sides are AB and DC, AD and BC ; the pairs of opposite angles are $\angle A$ and $\angle C$, $\angle B$ and $\angle D$; and the diagonals are AC and BD. In symbols, quadrilateral ABCD is written as $\square ABCD$.

To understand this further, let us revisit the following terms and their definitions:

Consecutive vertices refer to vertices that are connected by line segments called sides. The vertices A and B, B and C, C and D, and D and A are pairs of consecutive vertices. Points B and D are vertices but not consecutively drawn.

Adjacent sides are two sides with a common endpoint. AB and BC, BC and CD, CD and DA, and DA and AB are pairs of adjacent sides.

Adjacent angles are two consecutive angles that share a common side. $\angle A$ and $\angle B$ share a common side AB, $\angle B$ and $\angle C$, $\angle C$ and $\angle D$, and $\angle D$ and $\angle A$ are pairs of adjacent angles.

Opposite sides are two sides that do not have a common endpoint. AB and DC are opposite sides, so are AD and BC.



Opposite angles are two angles which do not have a common side. $\angle A$ and $\angle C$ are opposite angles, so are $\angle B$ and $\angle D$.

Quadrilaterals can be classified according to shape.

A quadrilateral with both pairs of opposite sides parallel is called a **parallelogram**.

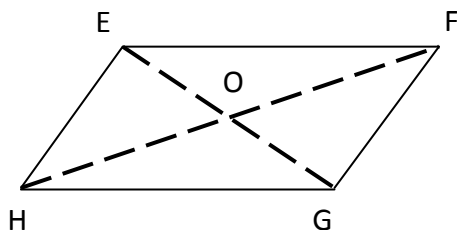
A quadrilateral with four right angles is called a **rectangle**.

A quadrilateral having four equal sides is called a **rhombus**.

A quadrilateral having four equal sides and four right angles is a **square**.

A quadrilateral with exactly one pair of opposite sides parallel is a **trapezium**.

The quadrilateral which has special characteristics that is usually studied first is the parallelogram because the rest, except the trapezium, are special parallelograms. As **defined**, a quadrilateral is a parallelogram if the two pairs of opposite sides are parallel. Examine the figure below.



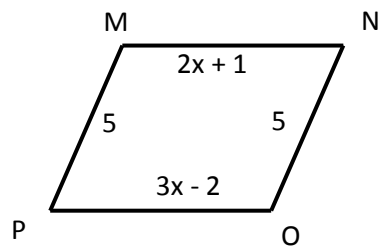
If quadrilateral EFGH is a parallelogram, then $EF \parallel HG$ and $EH \parallel FG$. We can then use what we know about parallel lines and the properties of parallelograms.

Properties of Parallelogram

1. The opposite sides of a parallelogram are equal and parallel.
If $EF \parallel HG$ and $EH \parallel FG$ in EFGH, then $EF = HG$, and $EH = FG$.
2. The opposite angles of a parallelogram are equal.
The pairs opposite angles of EFGH are $\angle E$ and $\angle G$, and $\angle F$ and $\angle H$.
Thus, $m\angle E = m\angle G$, and $m\angle F = m\angle H$.
3. Consecutive angles of a parallelogram are supplementary.
 $\angle E$ and $\angle F$, $\angle F$ and $\angle G$, $\angle G$ and $\angle H$, $\angle H$ and $\angle E$ are pairs of consecutive angles.
Therefore, $\angle E + \angle F = 180$, and so are the rest of the angle pairs.
4. The diagonals of a parallelogram bisect each other.
The diagonals EG and FH of EFGH intersect at point O. Hence, if point O is the midpoint of the two diagonals, then $EO = OG$ and $HO = OF$.



Example 1 What value of x will make each quadrilateral a parallelogram?



Solution

Using the first property for having two pairs of opposite sides parallel and equal,

Given: $MP = 5$, $NO = 5$

$MN = 2x + 1$, $PO = 3x - 2$

If $MP \parallel NO$ and $MP = NO$, we have $MN \parallel PO$ and $MN = PO$

For $MN = PO$

$$2x + 1 = 3x - 2$$

$$2x - 3x = -2 - 1$$

$$-1x = -3$$

$$\mathbf{x = 3}$$

If $x = 3$,

$$MN = 2x + 1$$

$$MN = 2(3) + 1$$

$$MN = 6 + 1$$

$$MN = 7$$

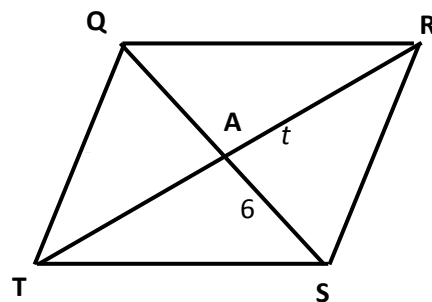
$$PO = 3x - 2$$

$$PO = 3(3) - 2$$

$$PO = 9 - 2$$

$$PO = 7$$

Since the opposite sides MN and PO are equal given that $MP = NO$, then $MN \parallel PO$. Therefore we conclude that quadrilateral $MNOP$ is a parallelogram for it produces two pairs of opposite side parallel and equal.



The diagonals $\overline{TR}$ and $\overline{QS}$ of quadrilateral $QRST$ bisect each other at point A .

If $\overline{QA} = \overline{AS}$, then $\overline{TA} = \overline{AR}$

Solution

$$\overline{TA} = \overline{AR}$$

$$2x + 5 = 5x - 1$$

$$2x - 5x = -1 - 5$$

$$-3x = -6$$

$$\mathbf{x = 2}$$

If $x = 2$

$$TA = 2x + 5$$

$$TA = 2(2) + 5$$

$$TA = 4 + 5$$

$$TA = 9$$

$$AR = 5x - 1$$

$$AR = 5(2) - 1$$

$$AR = 10 - 1$$

$$AR = 9$$



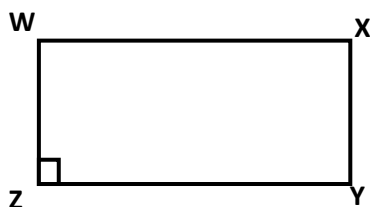
Since the diagonals TR and QS bisect each other, we conclude that the quadrilateral is a parallelogram.

We have just studied the properties of a parallelogram. All the properties of a parallelogram are also true for rectangle, square and rhombus which are special parallelograms. Special parallelograms have also their own unique properties in addition to these properties.

Any parallelogram with one right angle is a **rectangle**

Since a rectangle is a parallelogram for it possesses all the properties of a parallelogram, we can prove that **all angles of a rectangle are right angles**.

Example: Given: WXYZ is a parallelogram and $\angle Z$ is a right angle.



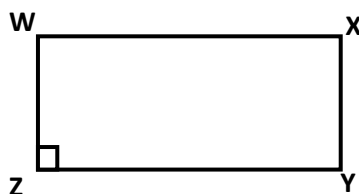
Prove: $\angle W = \angle X = \angle Y = \angle Z = 90^\circ$

Solution: Since a rectangle is a parallelogram, opposite angles are equal according to one of the properties of a parallelogram. If $m\angle Z = 90$, then $m\angle X = 90$. Another property is that consecutive angles of a parallelogram are supplementary.

$\angle W$ and $\angle Z$ are consecutive angles of the given parallelogram and so are $\angle Z$ and $\angle Y$. If $\angle W + \angle Z = 180^\circ$ and $\angle Z = 90^\circ$, then $\angle W = 90^\circ$.

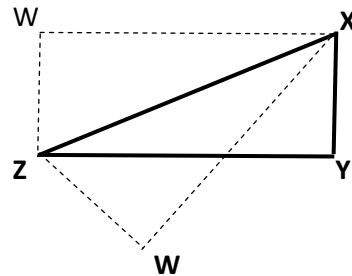
By using two of the properties of a parallelogram, we have proven that a rectangle has four right angles.

Now draw the diagonals WY and XZ and measure them. What do you find?





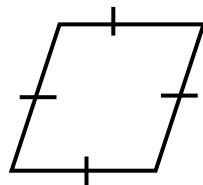
Or you can take identical rectangular sheets of paper then fold along a diagonal and compare its length to the diagonal of another paper as shown below. What do you notice?



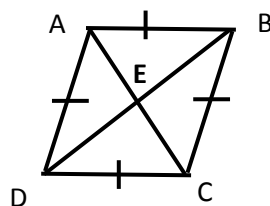
Your observations and conclusion may lead us to another unique property of a rectangle, that is, **the diagonals of a rectangle are equal**. This property proves that **if the diagonals of a parallelogram are equal, then the parallelogram is a rectangle**.

Another special parallelogram is a rhombus.

A parallelogram with four equal sides is a **rhombus**.



Using one of the properties of a parallelogram, that opposite sides are equal, we can prove the property of having four equal sides in a rhombus.

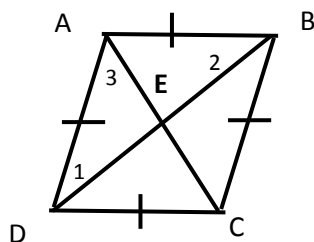


If we draw diagonals AC and BD at point E as their intersection and use the properties of a parallelogram, we will discover that the diagonals of a rhombus bisect each other. One unique characteristic of a rhombus is that its diagonals **are perpendicular to each other**.

Notice that diagonal BD bisects angles B and D, and using a similar approach, we can show that AC bisects angles A and C. Thus, **the diagonals of a rhombus bisect the four angles of the rhombus**.



Example



Parallelogram ABCD in the figure is a rhombus and $\angle 1 = 40^\circ$. Find the measure of each of the following angles.

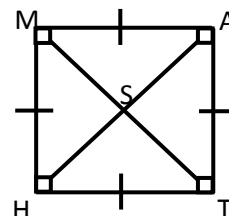
- a. $\angle 2$ b. $\angle 3$ c. $\angle DCB$

Solution

- a. Triangle ABC is isosceles since $AB = BC$. Hence, the base angles of the triangle must be equal. That is $\angle 1 = \angle 2 = 40^\circ$.
- b. In $\triangle DEA$, $\angle DEA$ is a right angle since the diagonals of a rhombus are perpendicular to each other. Since the sum of the measures of the angles of a triangle is 180° , the measure of $\angle 3$ must be 50° .
- c. The diagonals of a rhombus bisect the angles of the rhombus. If $\angle 3 = 50^\circ$, then $\angle DAB = 100^\circ$. Since opposite angles of a rhombus are equal in measure, then $\angle DCB$ must also be 100° .

Remember that a rhombus is not necessarily a rectangle since it may or may not contain four right angles. A rectangle is not necessarily a rhombus since it may or may not contain four equal sides. Thus, a square combines the properties of a rectangle with the properties of a rhombus, that is, **the diagonals of a square are equal, bisect its opposite angles, and intersect at right angles.**

Parallelogram MATH possesses the combined properties of a rectangle and a rhombus. It has four right angles and four equal sides formed by two pairs of parallel and congruent sides. The diagonals that are perpendicular and congruent bisect each other at point S.



The most special of all parallelograms is the **square**. A square is an **equilateral rectangle**. Since the definition uses the word equilateral, we can say that a square is also a rhombus.



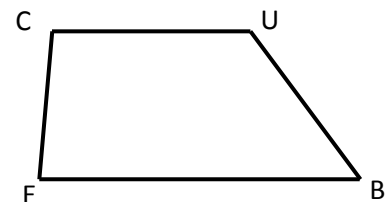
The table below will help you to learn the properties of various parallelograms.

Property	Parallelogram	Rectangle	Rhombus	Square
1. Opposite sides are $//$	✓	✓	✓	✓
2. Opposite sides are $\cong$	✓	✓	✓	✓
3. Opposite angles are $\cong$	✓	✓	✓	✓
4. Diagonal for two $\cong$ angles	✓	✓	✓	✓
5. Diagonal bisect each other	✓	✓	✓	✓
6. All angles are right angles		✓		✓
7. Diagonals are $\cong$		✓		✓
8. Diagonals are $\perp$			✓	✓
9. Diagonals bisect opposite angles			✓	✓
10. All sides are $\cong$			✓	✓

A quadrilateral with fewer special properties as compared to parallelogram but has interesting properties of its own is the trapezoid.

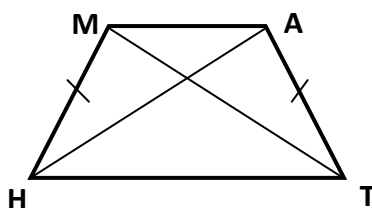
A **trapezium** is a quadrilateral with only one pair of parallel sides

The definition states that if CUBE is a trapezoid, and $CU // EB$, then CE is not parallel to UB. The parallel sides of a trapezoid are called its **bases** and the non parallel sides are its **legs**. The angles at the end of a base are called **base**



$\angle E$ and $\angle B$ are one pair of base angles as well as $\angle C$ and $\angle U$. If the legs are equal, then the trapezoid is isosceles.

An **isosceles trapezium** is one in which the nonparallel sides are equal.



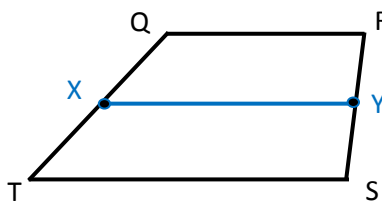
If we will compare trapezoids CUBE and MATH, the latter has a pair of equal sides but not the parallel sides. Notice that both pairs of base angles $\angle M$ and $\angle A$ and $\angle H$ and $\angle T$ are congruent.

The trapezoidal CUBE has no special property aside from having one pair of parallel sides. However, an isosceles trapezoid has some special properties, one of which is that **the base angles of a trapezoid are equal**.

We will learn another property of an isosceles trapezoid through the diagonals just like what we learnt on parallelograms. The diagonals will not bisect each other just like a parallelogram so we can conclude that the diagonals of an isosceles trapezoid are equal.

Consider the following experiment:

1. Draw any trapezoid QRST.
2. Locate the midpoints of legs QT and RS, then name the midpoints as point X and point Y.
3. Draw a segment joining the midpoints X and Y.
4. Note that XY appears to be parallel to TS and QR.
Use a protractor to check if $\angle STQ$ and $\angle YXQ$ are equal to support this observation.
5. Using a ruler, measure QR, TS and XY.
6. Note that XY appears to be the average of the measures of QR and TS.
That is, $XY = \frac{1}{2}(QR + TS)$.



Based from the experiment, the segment connecting the midpoints X and Y forms the median XY of trapezoid QRST.

The **median** of a trapezium is the segment joining the midpoints of the legs.

Example Quadrilateral ABCD is a trapezoid with XY as the median. Find the measure of the median XY.



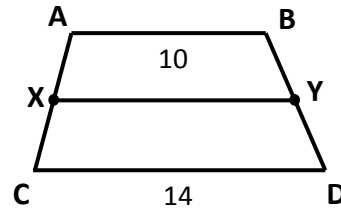
Solution

$$XY = \frac{1}{2}(AB + DC)$$

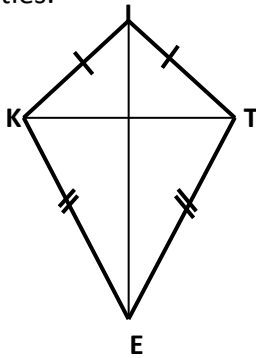
$$XY = \frac{1}{2}(10 + 14)$$

$$XY = \frac{1}{2}(24)$$

$$XY = 12$$



Another kind of quadrilateral is a kite. Refer to the figure below and discover the following properties.



1. It has two pairs of congruent sides: one pair of shorter adjacent sides and the other pair, of longer adjacent sides.
2. It has only one pair of congruent angles.
3. Only one diagonal determines two non-congruent isosceles triangles.
4. The diagonals are perpendicular.
5. Only one diagonal determines two congruent

**LEARNING ACTIVITY 11.4.1.4**

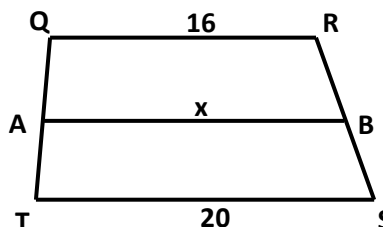
10 minutes

1. Write whether the statement is **TRUE** or **FALSE**.

- _____ a. A square is a parallelogram.
- _____ b. The diagonals of a rhombus are perpendicular.
- _____ c. A trapezoid is a parallelogram.
- _____ d. A square is a rhombus.
- _____ e. A square is a rectangle.
- _____ f. A rhombus is a square.
- _____ g. The diagonals of a rhombus are congruent.
- _____ h. The consecutive angles of a quadrilateral are congruent.
- _____ i. The consecutive angles of a rhombus are supplementary.
- _____ j. A quadrilateral is a parallelogram if both pairs of opposite angles are congruent.

2. Find the value of x .

Quadrilateral QRST is a trapezium.
AB is the median. If $QR = 16$ and $TS = 20$,
find x .



3. List two properties of a parallelogram

4. Which of the following statements is true, based on their properties: (i) a rectangle is a parallelogram (ii) a parallelogram is a rectangle. State your reason.

5. Name the quadrilateral that always has its diagonals intersecting at 90° .



11.4.1.5 Polygons

We have studied triangles and quadrilaterals and learnt that a triangle is a polygon consisting of three sides. Now, we will study polygons in general. The properties of a polygon are as follows:

1. A polygon is formed by segments, where no two adjacent segments form a straight angle.
2. A polygon is a closed plane figure.
3. A polygon is contained in one plane.
4. Each side of a polygon intersects only two sides.
5. The sides of a polygon intersect only at their endpoints.

The word polygon comes from two Greek words, “poly” which means many and “gonia” which means sides. The four important terms related to polygons are interior angle, side, vertex and diagonal. The segments that form a polygon are called the sides while the angles formed by the sides are called interior angles with their corresponding vertices.

A segment that joins a pair of nonadjacent vertices is called a diagonal. We can name a triangle with consecutive vertices. Likewise, we can name a polygon by its consecutive vertices.

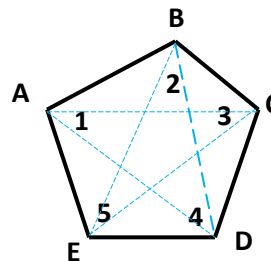
Given: Polygon ABCDE

Vertices: A, B, C, D, and E

Sides: $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, and $\overline{EA}$

Interior angles: Angles 1, 2, 3, 4, and 5

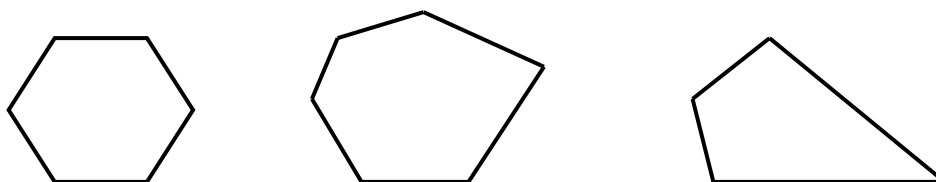
Diagonals: $\overline{AC}$, $\overline{AD}$, $\overline{BD}$, $\overline{BE}$, and $\overline{CE}$



The angles 1, 2, 3, 4, and 5 can be renamed as $\angle EAB$, $\angle ABC$, $\angle BCD$, $\angle CDE$ and $\angle DEA$ or reverse order of these letters.

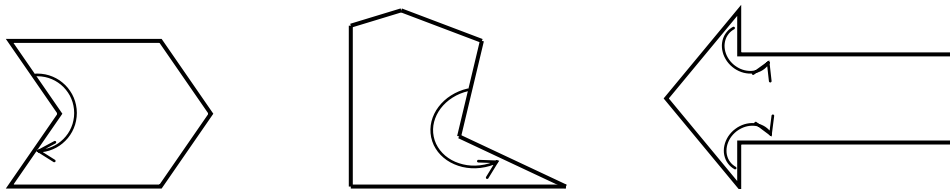
From the given figure, it is clear that there exists a relationship between the number of sides of a polygon and the number of diagonals that can be drawn from it.

Polygons can also be classified as convex or concave polygon. A polygon is **convex** if each angle is less than 180° . The following are examples of convex polygons:





A polygon is **concave** or **re-entrant** if at least one interior angle is a reflex angle. The following are examples of convex or re-entrant polygons:



We classify polygons according to the number of sides.

Prefix	Number of sides (n)	Name of Polygon
tri-	3	triangle
quadri-	4	quadrilateral
penta-	5	pentagon
hexa-	6	hexagon
hepta-	7	heptagon
octa-	8	octagon
nona-	9	nonagon
deca-	10	decagon
hendeca-	11	hendecagon
duodeca-	12	duodecagon
	n	n-gon

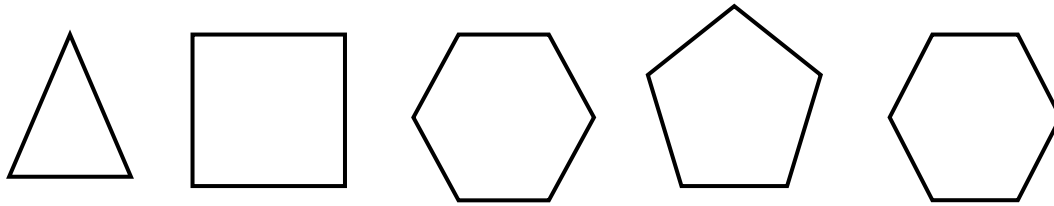
We call triangles having equal sides as equilateral triangles. Polygons with equal sides are also called **equilateral** and polygons with equal angles, **equiangular**.

Note that not all equilateral polygons are equiangular. We know that a rectangle is equiangular but not equilateral. A rhombus may be equilateral but not equiangular. On the other hand, a square is both equilateral and equiangular. Thus, a square is a regular polygon.

A polygon that is both equilateral and equiangular is a **regular polygon**. All other polygons are said to be **irregular polygons**.

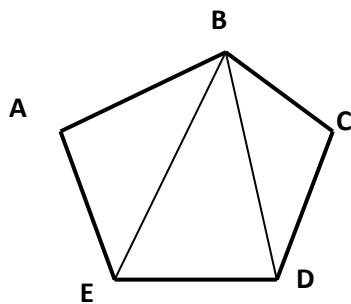


Examples of regular polygons



We know that the sum of all interior angles of a triangle is 180° and that the sum of all interior angles of a quadrilateral is 360° . Let us now **find** the sum of all interior angles of polygons.

Given a pentagon ABCDE, find the sum of all the angles.



If we divide the pentagon into three smaller triangles, we have:

$$\triangle ABE = 180^\circ$$

$$\triangle BED = 180^\circ$$

$$\triangle BCD = 180^\circ$$

The sum of the angles of the pentagon = 540° .

We may conclude that the sum of the angles of any polygon is obtained by dividing the polygons into triangles, and then multiply the result by 180° since the sum of all angles in a triangle is 180° .

There is a relationship between the number of sides and the least number of triangles obtained as in polygon ABCDE. The number of triangles is always 2 less than the number of sides. If we define that the number of sides to be n , then the number of triangles inscribed in the polygon is $n - 2$.

So to find the sum of interior angles of polygons (S) we multiply the number of triangles ($n - 2$) by 180° , thus the formula **$S = 180^\circ(n - 2)$** .

Example What is the sum of interior angles of an icosagon?

Solution

Icosagon = 20 sides

$$\begin{aligned} S &= 180^\circ (n - 2) \\ &= 180^\circ (20 - 2) \\ &= 180^\circ (18) \\ &= 3240^\circ \end{aligned}$$

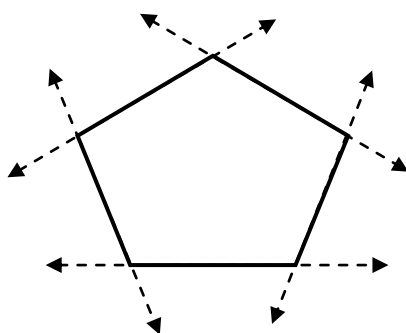


Number of sides (n)	Polygon	Number of triangles	Sum of interior angles	Interior of regular	Exterior of regular
3	triangle	$3 - 2 = 1$	180°	60	120
4	quadrilateral	$4 - 2 = 2$	$2 \cdot 180^\circ = 360^\circ$	90	90
5	pentagon	$5 - 2 = 3$	$3 \cdot 180^\circ = 540^\circ$	108	72
6	hexagon	$6 - 2 = 4$	$4 \cdot 180^\circ = 720^\circ$	120	60
7	heptagon	$7 - 2 = 5$	$5 \cdot 180^\circ = 900^\circ$	$128\frac{4}{7}$	$51\frac{3}{7}$
8	octagon	$8 - 2 = 6$	$6 \cdot 180^\circ = 1080^\circ$	135	45
9	nonagon	$9 - 2 = 7$	$7 \cdot 180^\circ = 1260^\circ$	140	40
10	decagon	$10 - 2 = 8$	$8 \cdot 180^\circ = 1440^\circ$	144	36
n	n-gon	$n - 2$	$(n - 2) \cdot 180^\circ = S$		

The sum (S) of the interior angles of a polygon is $S = 180^\circ (n - 2)$ where n = the number of sides.

We can also find the sum of the measures of the exterior angles of any convex polygon. The figure of a regular pentagon below shows that there are two exterior angles at each vertex. At any given vertex, the two exterior angles are vertical angles which may be concluded as congruent.

However, when we obtain the sum of the measures of exterior angles of a polygon, we consider only one of the exterior angles. It is not important which exterior angle we include in finding the sum of the measures.



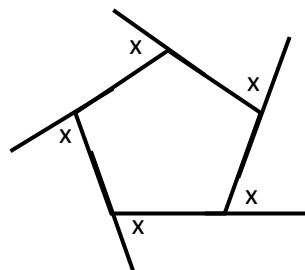
As shown in the figure, any interior angle and its corresponding exterior angle forms a linear pair. A linear pair is formed by two adjacent angles that are supplementary. If the sum of



the angles of a regular pentagon is 540° , then $540^\circ \div 5 = 108^\circ$. This means that each interior angle measures 108° . Since interior and exterior angles are supplementary, therefore each exterior angle of the regular pentagon measures 72° .

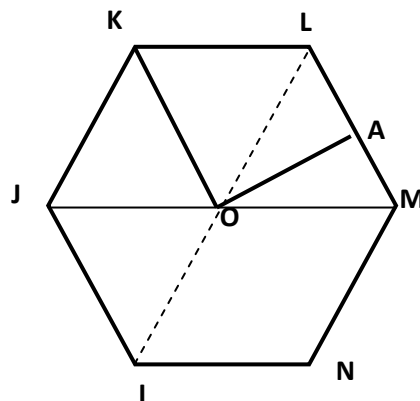
Thus, we can conclude that the sum of the measures of the exterior angles of a regular pentagon is 360° , that is, $72^\circ \cdot 5 = 360^\circ$. If we repeat this procedure with a hexagon and a heptagon, we arrive at this conclusion.

The sum of the exterior angles of any polygon is 360° .



The shape is a regular pentagon. Since all interior angles are equal, all exterior angles will also be equal. Thus, $x + x + x + x + x = 360^\circ$

Investigating a regular Hexagon

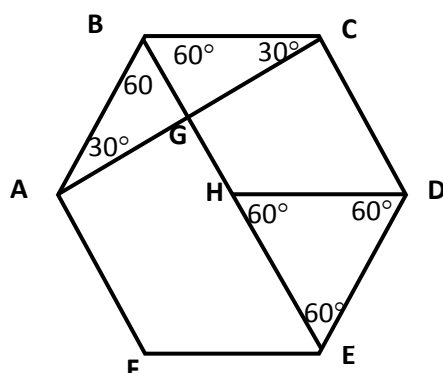


Regular Hexagon

OK is the long radius
OA is the short radius

OA in the old use is referred to as an APOTHEM

Let us find out what angles and shapes can we obtain when we rule diagonals and long radii.



ABC, BCD, CDE, DEF, EFA and FAB are isosceles triangles.

DEH, CDH, BCH, ABH, FAH and EFH are equilateral triangles.

BGC and BAG are a right triangles.

CDHG and AGEF are trapeziums.

**LEARNING ACTIVITY 11.4.1.5**20 minutes

1. Find the sum of the interior angles of the following polygons:
 - a. undecagon
 - b. dodecagon

 2. Find the measure of each interior angle of
 - a. a regular octagon
 - b. a regular decagon

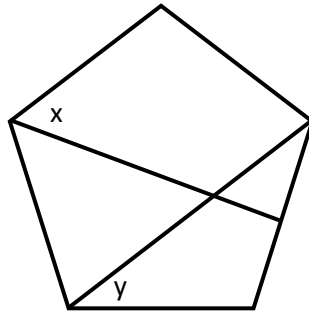
 3. Find the value of an exterior angle of
 - a. a regular triangle
 - b. a regular pentadecagon (15 sides)

 4. Five of the interior angles of a hexagon are 100° , 110° , 125° , 134° , and 140° . Calculate the measure of the other interior angle.

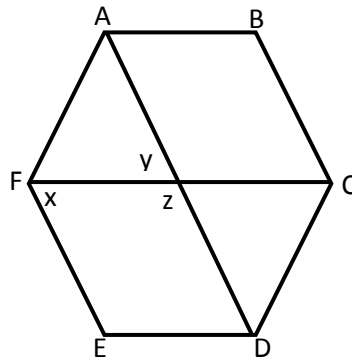
 5. Calculate interior angles of a regular icosagon (20 sides).
-



6. Below is a regular pentagon. Find values of angles marked x and y .

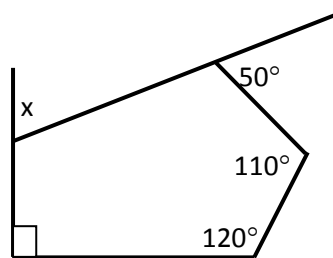


7. Solve for angles x , y and z in the regular hexagon ABCDEF.

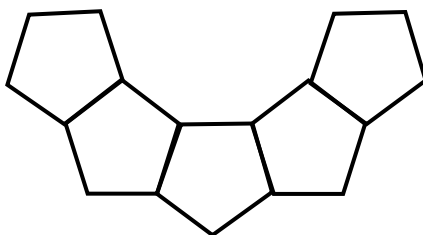


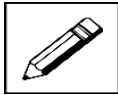
8. Find area of octagon with sides of 3cm.

9. Solve for x .



10. What is the total area covered if area of a pentagon is 3.82cm^2 ?



**SUMMATIVE TASK 11.4.1**

20 minutes

I. Multiple Choice: Write the letter of your choice before each number.

_____ 1. What is Geometry?

- a. Geometry is the study of earth.
- b. Geometry is the study of earth measurement.
- c. Geometry is the study of plane and solid figures.
- d. Geometry is the study of triangle measurement.

_____ 2. Who among the following is the father of Geometry?

- a. Pythagoras
- b. Euclid
- c. Rene Descartes
- d. Blaise Pascal

_____ 3. A simple closed plane figure made up of line segment

- a. point
- b. line
- c. plane
- d. polygon

_____ 4. What is a seven-sided polygon?

- a. pentagon
- b. heptagon
- c. hexagon
- d. octagon

_____ 5. A quadrilateral that has one pair of parallel side

- a. general quadrilateral
- b. parallelogram
- c. trapezoid
- d. trapezium

_____ 6. Which angle pair has a sum of 180° ?

- a. complementary angles
- b. supplementary angles
- c. vertical angles
- d. adjacent angles

_____ 7. Which of the following is the symbol for parallelism?

- a. $\cong$
- b. $\perp$
- c. $//$
- d. $\times$

_____ 8. It refers to the total length of the sides of a polygon

- a. area
- b. volume
- c. circumference
- d. perimeter

_____ 9. A two-dimensional figure that has four right angles and four equal sides

- a. quadrilateral
- b. square
- c. parallelogram
- d. rectangle



- _____ 10. The longest side of a right triangle is called
- a. altitude b. base c. hypotenuse d. height
- _____ 11. In which of the following can an angle be named?
- I. using the vertex III. using a numeral
II. using three letters IV. using a small letter
- a. I and II only b. I and III only
c. I, II and III only d. all of the above
- _____ 12. Points that lie on the same line are
- a. linear pair b. collinear c. coplanar d. linearity
- _____ 13. What geometric figure is represented by the polished surface of a mirror?
- a. polygon b. plane c. quadrilateral d. rectangular
- _____ 14. Which one refers to a ray/line segment that divides the angle into congruent angles?
- a. angle divisor c. half angle
b. angle bisector d. perpendicular angle
- _____ 15. A line segment drawn from one vertex of a triangle perpendicular to its opposite Side
- a. median c. angle bisector
b. altitude d. perpendicular bisector
- _____ 16. a two dimensional figure with two bases?
- a. square b. rectangle c. parallelogram d. trapezoid
- _____ 17. What is the least number of points needed to form a plane?
- a. 1 b. 2 c. 3 d. 4
- _____ 18. What kind of angle is opposite the hypotenuse?
- a. acute angle b. obtuse angle c. reflex angle d. right angle
- _____ 19. A polygon is called _____ if and only if the lines containing the sides do not contain points in the interior of a polygon.



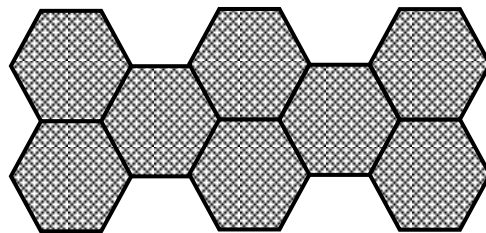
- a. concave b. convex c. non-convex d. n-gon

____ 20. A part of a line with one excluded endpoint is

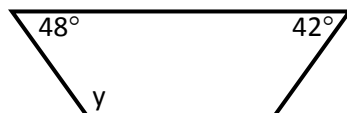
- a. ray b. half ray c. line segment d. angle

II. Problem Solving

1. A regular octagon has sides of 4cm. What is the perimeter of the regular octagon?
2. Calculate total area covered by the regular polygon given that area of one is 2.8cm^2 .



3. Find the value of angle y .

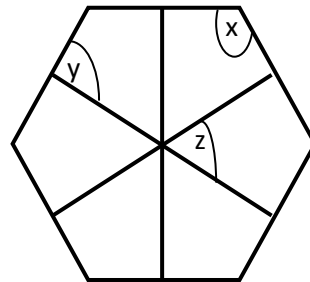


4. Show that the exterior angle of a regular decagon is 36° . All required work must be shown.
5. Use the formula $S = 180^\circ(n - 2)$ to find sum of interior angles of a nonagon.

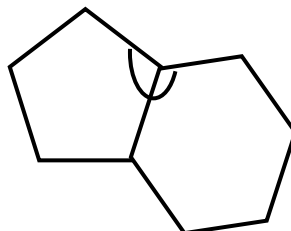


6. A regular polygon has exterior angle of 22.5° . How many sides has the polygon? Hint: Sum of exterior of all polygons is 360° .

7. The shape below is a regular hexagon. Study the division by apothems (or short radii) and calculate angles marked with letters.



8. Below is a compound shape form by combination of a regular pentagon and a regular hexagon. Find the angle made by them indicated by the arc.





11.4. 2: CONGRUENCY AND SIMILARITY

We can see different objects with triangular figures around us maybe because their shape is easy to handle and manipulate into different designs. Roofs, bridges, and even early warning devices are triangular in shape.

Likewise, the idea of congruence is all around us. Any two geometric figures having the same size and shape are called congruent and it is denoted by the symbol $\cong$. The symbol $\sim$ indicates the same shape and is used as the sign for similarity. The symbol $=$ indicates the same size and is used as a sign for equality.

It always helps to recognize congruent figures in the same orientation. When two figures are congruent, you may slide, flip, or rotate them until they overlap exactly.

Congruency, like equality, represents equivalent relations since it has reflexive, symmetric and transitive properties which may be used in proving congruent triangles.

Property	Relations expressed	Symbols
Reflexive	Anything is congruent to itself.	$\overline{AB} \cong \overline{AB}$
Symmetric	The members on either side of congruent symbol may be interchanged.	If $\angle A \cong \angle B$, then $\angle B \cong \angle A$
Transitive	If two quantities are congruent to the same quantity, then they are congruent to each other.	If $\overline{AB} \cong \overline{CD}$ and $\overline{XY} \cong \overline{CD}$, then $\overline{AB} \cong \overline{XY}$.

A **proof** is a sequence of true statements placed in logical order. To show the validity of the proof, a reason that justifies each statement must be provided. The following can be used in justifying statements.

Inductive reasoning is the process of gathering specific information obtained through observation and measurement and then making a conjecture based on the gathered information. A **conjecture** is a conclusion made from observing data.

Deductive reasoning is the process of showing that certain statements follow logically from agreed upon assumptions and proven facts. Deductive reasoning is used in formal geometric proofs and is often used in proving.



11.4.2.1: Congruent Triangles

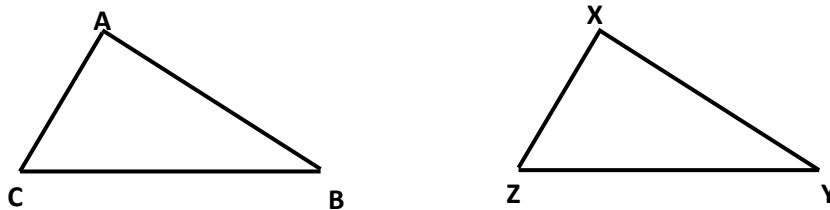
You have learnt previously about congruent segments and congruent angles. This time, you will learn about congruent triangles. To define, we need six pairs of corresponding parts between two triangles to establish their congruence. In general, two triangles are congruent if their vertices can be paired off so that corresponding angles are congruent and corresponding sides are congruent.

CPCTC (Corresponding Parts of Congruent Triangles are Congruent)

If two triangles are congruent, then the six pairs of corresponding parts are congruent.

Corresponding angles are pairs of angles whose vertices are paired in a given correspondence between two triangles.

Corresponding sides are pairs of segments whose endpoints are vertices that are paired in a given correspondence between two triangles. Corresponding sides of triangles lie opposite corresponding angles.



We can speak of a correspondence between the triangles as $\triangle ABC \leftrightarrow \triangle XYZ$. We may pair the vertices as $A \leftrightarrow X$, $B \leftrightarrow Y$ and $C \leftrightarrow Z$. Because A and X are corresponding vertices, $\angle A$ and $\angle X$ are corresponding angles. Other corresponding angles are $\angle B$ and $\angle Y$, and $\angle C$ and $\angle Z$. Also, AB and XY are corresponding sides like $AC \leftrightarrow XZ$ and $BC \leftrightarrow YZ$.

When the following six conditions are true for $\triangle ABC$ and $\triangle XYZ$, then the triangles are congruent.

Therefore, $\triangle ABC \cong \triangle XYZ$ iff

$$\angle A \cong \angle X,$$

$$\angle B \cong \angle Y,$$

$$\angle C \cong \angle Z,$$

$$\overline{AB} \cong \overline{XY},$$

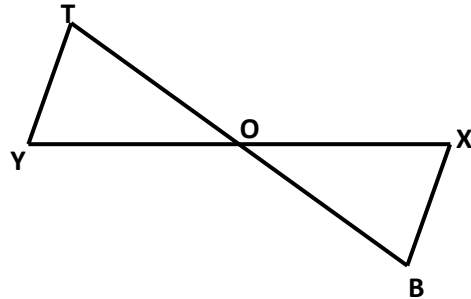
$$\overline{CB} \cong \overline{YZ}, \text{ and}$$

$$\overline{AC} \cong \overline{XZ}.$$



Example The two triangles in the figure are congruent. Complete the following statements:

- $\triangle TOY \cong$ _____
- $\angle T \cong$ _____ because _____.
- Since _____ = XO, O is the midpoint of _____.
- $\angle Y \cong$ _____ because _____.



Solution

- $\triangle BOX$
- $\angle B$, because corresponding parts of congruent triangles are congruent.
- YO, because O is the midpoint of XY.
- $\angle X$, because corresponding parts of congruent triangles are congruent.

In proving triangle congruence, general statements that we accept as starting points are postulates and theorems.

Postulates are statements that are accepted as true without proof.
Theorems are statements that need to be proven.

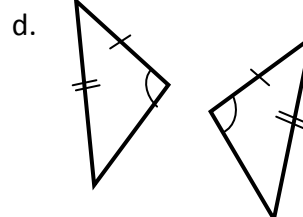
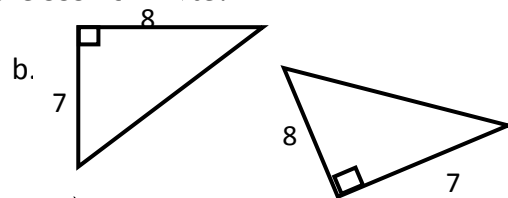
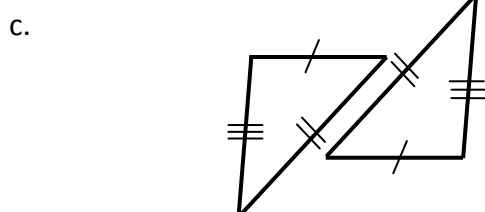
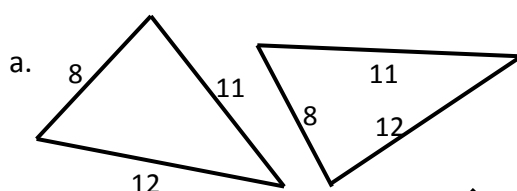
11.4.2.2 Proving the Congruency of Triangles

There are several shortcut methods in proving triangle congruence than by just demonstrating that they agree in six pairs of the parts of triangles. Hence, this suggests the following postulates:

Side-Side-Side or SSS Postulate

If three sides of one triangle are equal to the corresponding sides of another triangle, then the two triangles are congruent.

Example Which pairs of triangles are congruent by the SSS Postulate?





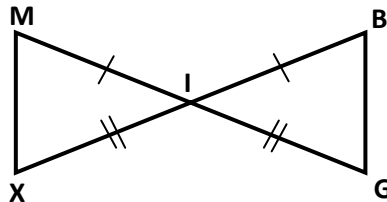
Pairs **a** and **c** are congruent by SSS Postulate. Note that the congruent parts of the two pairs of triangles on **c** and **d** are indicated by similar markings.

In addition to SSS Postulate, another shortcut method to prove triangle congruence is the Side-Angle-Side Postulate. In comparing two triangles, the pair of equal angles must be sandwiched in between the two pairs of congruent sides.

Side-Angle-Side or SAS Postulate

If two sides and the included angle of one triangle are congruent to the corresponding parts of another triangle, then the two triangles are congruent.

The pairs of triangles (b) on page 44 illustrate an SAS Postulate. Another example is shown below.

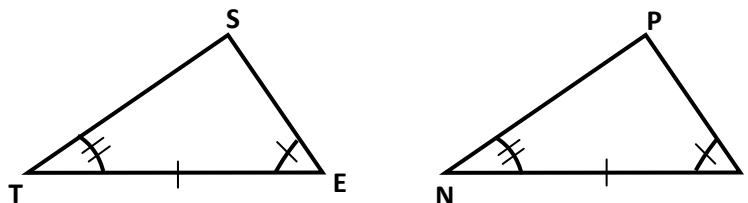


From the given figure, the first pair of congruent sides indicated by the markings is MI and BI. The included angles are $\angle MIX$ and $\angle BIG$ which form vertical lines and therefore are congruent. The last pair of corresponding and congruent sides is XI and GI. We therefore conclude that ΔMIX and ΔBIG are congruent by SAS Postulate.

Similarly, if two angles and the included side of one triangle are equal to the corresponding parts of another triangle, then the triangles are congruent.

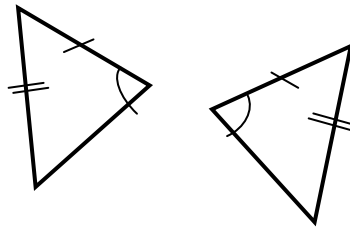
Angle-Side-Angle or ASA Postulate

If two sides and the included angle of one triangle are congruent to the corresponding parts of another triangle, then the two triangles are congruent.



$\Delta SET \cong \Delta PIN$ since the pair of congruent sides TE and NI are sandwiched by the two pairs of corresponding and congruent angles $\angle T$ and $\angle E$, and $\angle N$ and $\angle I$.

However, we cannot prove that two triangles are congruent if two sides of one triangle are equal to two sides of another triangle and an angle that is not included is equal to the corresponding angle but also not included of another triangle. This is illustrated in figure d on page 44.



To prove congruency of triangles, the following will serve as your guide.

1. Mark the diagram with the Given.
2. Mark the additional parts that are equal such as vertical angles, sides or angles shared by both triangles.
3. Label angles with numbers to make the writing proof easier.
4. Decide which method of proving triangles will be used.
5. Write a formal proof. In the two-column proof, number each statement to be written in the left column and opposite each statement give the reason to support it in the right column. The reason may be any or all of the following:

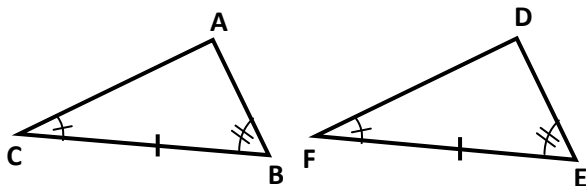
- | | |
|-----------------|----------------|
| a. given | c. a postulate |
| b. a definition | d. a theorem |

Often, there are different ways of proving what is required. If you did not use any or all of the given conditions, your proof is not be correct.

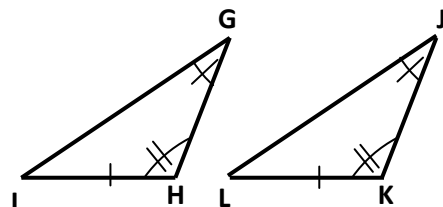
Angle-Angle-Side (AAS) or Side-Angle-Angle (SAA) Theorem

If two consecutive angles and a side opposite one of the given angles are congruent to two angles and the corresponding side opposite one of the angles, then the two triangles are congruent. AAS or SAA is a theorem because it can be proven using the other triangle congruence postulates.

Example 1 Classify the triangles as ASA Postulate or AAS Theorem.



Solution: ASA Postulate



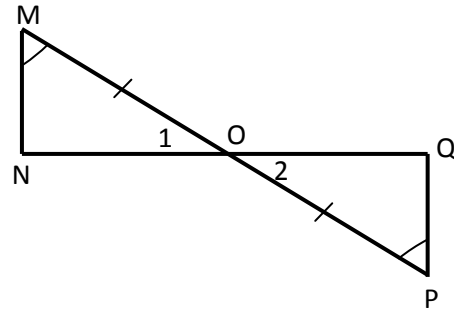
AAS Theorem



Example 2. Write a two-column proof.

Given: O is the midpoint of $\overline{MP}$
 $\angle M \cong \angle P$

Prove: $\triangle MON \cong \triangle POQ$



Plan for proof: Since the figure shows some congruent parts for $\angle M \cong \angle P$, $\overline{MO} \cong \overline{PO}$ and $\angle 1 \cong \angle 2$, prove that the two triangles are congruent using ASA Postulate.

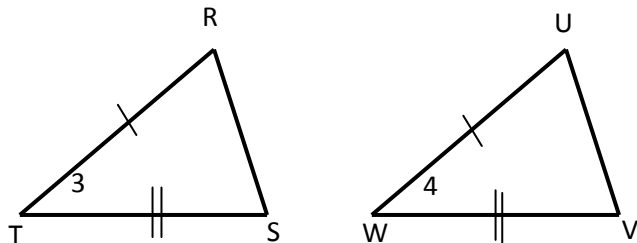
Proof:

Statements		Reasons
1) $\angle M \cong \angle P$	Angle	1) Given
2) O is the midpoint of MP		2) Given
3) $\overline{MO} \cong \overline{PO}$	Side	3) Definition of midpoint of a segment.
4) $\angle 1 \cong \angle 2$	Angle	4) Vertical angles are congruent.
5) $\triangle MON \cong \triangle POQ$		5) ASA Postulate

Example 3 Write a two-way proof.

Given: $m\angle 3 = m\angle 4$
 $TR \cong WU$; $TS \cong WV$

Prove: $\triangle RST \cong \triangle UVW$



Plan for proof: Use $TR \cong WU$, $m\angle 3 = m\angle 4$ and $TS \cong WV$ as statements in this order to prove that $\triangle RST \cong \triangle UVW$.

Proof:

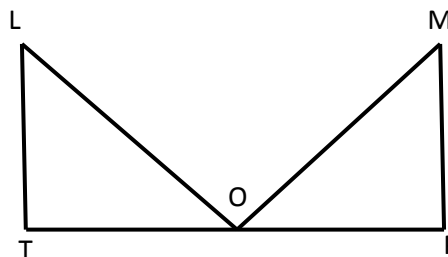
Statements	Reasons
1) $TR \cong WU$	1) Given
2) $m\angle 3 = m\angle 4$	2) Given
3) $\angle 3 \cong \angle 4$	3) Definition of congruent angles.
4) $TS \cong WV$	4) Given
5) $\triangle RST \cong \triangle UVW$	5) SAS Postulate



Example 4 Write a two-way proof.

Given: $LT \perp TP$, $MP \perp TP$
 O is the midpoint of TP
 $LT \cong MP$

Prove: $\Delta LTO \cong \Delta MPO$



Plan for proof: Since the figure does not have symbols to show relationship between ΔLTO and ΔMPO , the basis in showing triangle congruence will be from the given. It is better to put markings on the figure to show congruent parts.

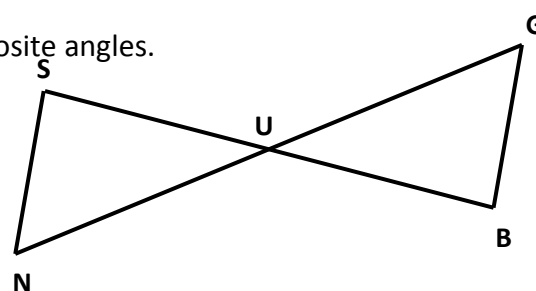
Proof:

Statements	Reasons
1) $LT \perp TP$, $MP \perp TP$	1) Given
2) $\angle T$ and $\angle P$ are right angles.	2) Definition of perpendicular.
3) $\angle T = \angle P$	3) All right angles are equal.
4) $\angle T \cong \angle P$	4) Definition of congruent angles.
5) O is the midpoint of TP	5) Given
6) $TO \cong PO$	6) Definition of midpoint.
7) $LT \cong MP$	7) Given
8) $\Delta LTO \cong \Delta MPO$	8) SAS Postulate

Example 5 Write a two-way proof.

Given: SB bisects NG
 $\angle SUN$ and $\angle BUG$ are vertically opposite angles.
 $\angle N \cong \angle G$

Prove: $AB \cong DC$



Proof:

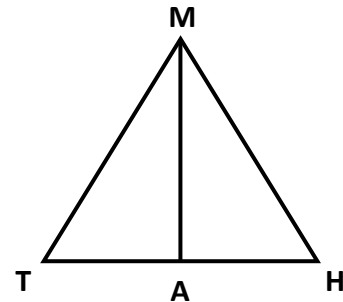
Statements	Reasons
1) SB bisects NG	1) Given
2) U is the midpoint of NG .	2) Definition of bisector of a segment.
3) $NU \cong UG$	3) Definition of midpoint.
4) $\angle SUN$ and $\angle BUG$ are vertically opposite angles.	4) Given
5) $\angle SUN \cong \angle BUG$	5) Vertical angles are congruent.
6) $\angle N \cong \angle G$	6) Given
7) $\Delta SUN \cong \Delta BUG$	7) SAA or AAS Theorem
8) $AB \cong DC$	8) CPCTC



Example 6 Write a two-way proof.

Given: $\triangle MHT$ is isosceles with TH as its base.
 MA bisects TH

Prove: $\triangle MAT \cong \triangle MAH$

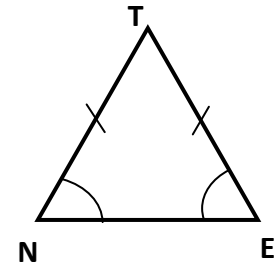


Proof:

Statements	Reasons
1) $\triangle MHT$ is isosceles with TH as its base.	1) Given
2) $MT \cong MH$	2) Definition of isosceles triangle.
3) MA bisects TH .	3) Given
4) $TA \cong AH$	4) Definition of bisector of a segment.
5) $MA \cong MA$	5) Reflexive Property
6) $\triangle MAT \cong \triangle MAH$	6) SSS Postulate

Previously, we learnt that two triangles are congruent. This lesson is actually an extension because we can now apply the definition of congruent triangles, that is, the “Corresponding Parts of Congruent Triangles are Congruent (CPCTC)”. Theorems on isosceles triangles can be proven using CPCTC.

Let us revise the parts of an isosceles triangle.



$\triangle TEN$ is an isosceles triangle.

TN and TE are the legs and NE is the base of $\triangle TEN$.

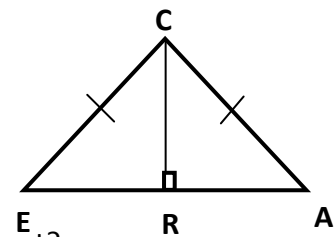
$\angle N$ and $\angle E$ are the base angles while $\angle T$ is the vertex angle.

Isosceles Triangle Theorem 1

If two sides of a triangle are congruent, then the angles opposite those sides (base angles) are congruent.

Example Given: $CE \cong CA$

Prove: Is $CR \cong CR$? Why?
Is $\angle CRE \cong \angle CRA$? Why?
What are the base angles?
How can we prove that the base angles are congruent?





Solution

$CR \cong CR$ Reflexive Property
 $\angle CRE \cong \angle CRA$ Right angles are congruent
 The base angles are $\angle E$ and $\angle A$.
 The base angles can be proven to be congruent by using CPCTC.

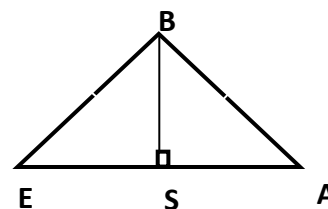
Suppose you are told that the two angles of a triangle are congruent. How can you conclude that the sides opposite the congruent angles are congruent? This leads us to another theorem on isosceles triangle.

Isosceles Triangle Theorem 2

If two angles of a triangle are congruent, then the sides opposite the congruent angles are congruent.

Example Given: $\triangle BAE$ with $\angle E \cong \angle A$
 A perpendicular bisector BS from B to EA .

Prove: Are angles BSE and BSA right angles? Why?
 Is $ES \cong SA$? Why?
 What postulate will show that $\triangle BSE \cong \triangle BSA$?



Solution

$\angle BSE$ and $\angle BSA$ are right angles.	Perpendicular lines form right angles.
$ES \cong SA$	Definition of perpendicular bisector
$BS \cong BS$	Reflexive Property
$\triangle BSE \cong \triangle BSA$	SAS Postulate

If $\triangle BSE \cong \triangle BSA$, we can conclude that $BE \cong BA$ using CPCTC.

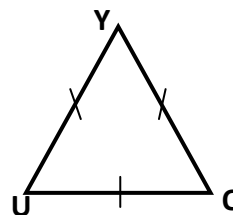
In any equilateral triangle, all three sides are equal. Thus, an equilateral triangle is an isosceles triangle and infer any of the three sides as the base. Since an equilateral triangle has three equal sides, we arrive with this theorem.

Equilateral Triangle Theorem

If a triangle is equilateral, then it is also equiangular.

Example Given: $\triangle YOU$ is an equilateral triangle.
 $YO \cong OU \cong YU$

Prove: $\triangle YOU$ is an equiangular triangle.





Proof:

Statements	Reasons
1) $YO \cong OU$	1) Given
2) $\angle U \cong \angle O$	2) Isosceles Triangle Theorem
3) $OU \cong YU$	3) Given
4) $\angle U \cong \angle Y$	4) Isosceles Triangle Theorem
5) $\angle Y \cong \angle O \cong \angle U$	5) Transitive Property

Since we have proven that the three angles are congruent, then $\triangle YOU$ is an equiangular triangle. If $\triangle YOU$ is both an equilateral and an equiangular triangle, then we can come up with another theorem about equiangular triangle which is the converse of equilateral triangle theorem.

Equiangular Triangle Theorem

If a triangle is equiangular, then it is also equilateral.

You have learnt about theorems on isosceles, equilateral and equiangular triangles and how to prove triangle congruence using SSS, SAS and ASA Postulates, and examples using SAA or AAS Theorem.

You will now learn how to prove some theorems involving right triangles and isosceles triangles. Recall that a right triangle is a triangle that contains one right angle. The right angle is sandwiched by the two sides called the legs while the other side which is the longest is called the hypotenuse. Let us now find shortcuts in proving triangle congruence between right triangles.

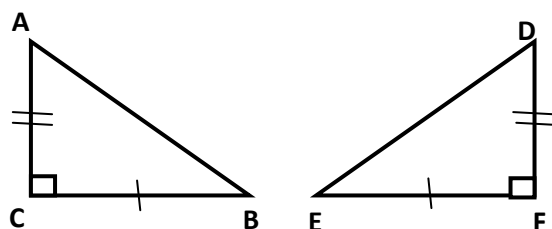
Leg-Leg (LL) Theorem

If the legs of one right triangle are congruent to the legs of another right triangle, then the two triangles are congruent.

Given: $\angle C$ and $\angle F$ are right angles.

$\overline{BC} \cong \overline{EF}$; $\overline{AC} \cong \overline{DF}$

Prove: $\triangle ABC \cong \triangle DEF$





Proof:

Statements	Reasons
1) $\overline{BC} \cong \overline{EF}$; $\overline{AC} \cong \overline{DF}$	1) Given
2) $\angle C$ and $\angle F$ are right angles.	2) Given
3) $\angle C \cong \angle F$	3) Right angles are congruent
4) $\triangle ABC \cong \triangle DEF$	4) SAS Postulate

Leg-Angle (LA) Theorem

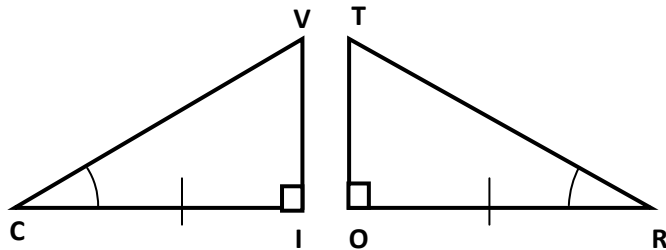
If a leg and an acute angle of one right triangle are congruent to the leg and an acute angle of another right triangle, then the two triangles are congruent.

Example Given: $\angle I$ and $\angle O$ are right angles.

$\overline{IC} \cong \overline{OR}$

$\overline{CV} \cong \overline{RT}$

Prove: $\triangle VIC \cong \triangle TOR$



Solution

By inspection of the figures alone, you can see that the two triangles can be guaranteed by the ASA Postulate.

Hypotenuse-Angle (HA) Theorem

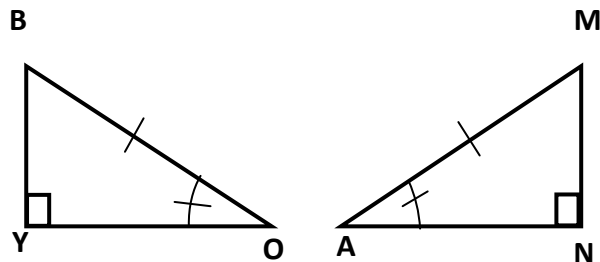
If the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and an acute angle of another right triangle, then the two triangles are congruent.

Example Given: $\angle Y$ and $\angle N$ are right angles.

$\overline{BO} \cong \overline{MA}$

$\angle O \cong \angle A$

Prove: $\triangle BOY \cong \triangle MAN$





Solution

Using SAA or AAS Theorem will guarantee that these two angles are congruent.

Hypotenuse-Leg (HL) Theorem

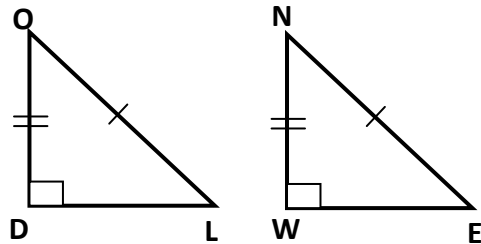
If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and corresponding leg of another right triangle, then the two triangles are congruent.

Example Given: $\angle D$ and $\angle W$ are right angle:

$$\overline{OL} \cong \overline{NE}$$

$$\overline{OD} \cong \overline{NW}$$

Prove: $\triangle OLD \cong \triangle NEW$



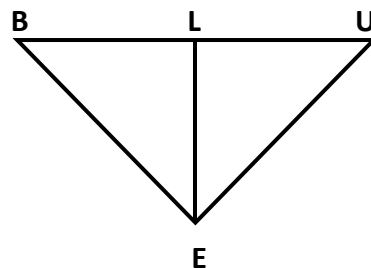
Solution

There is no SSA Postulate for triangles so the HL Theorem may be used if the triangles are right triangles. Since $\triangle OLD$ and $\triangle NEW$ are right triangles, then the two triangles are congruent.

Example Use a right triangle theorem in proving the statement.

Given: $\overline{EL}$ is the $\perp$ bisector of $\overline{BU}$.

Prove: $\triangle BEL \cong \triangle UEL$



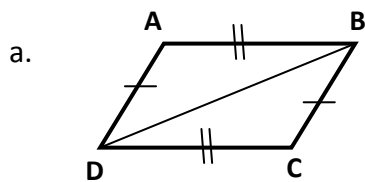
Proof:

Statements	Reasons
1) $\overline{EL}$ is the $\perp$ bisector of $\overline{BU}$.	1) Given
2) L is the midpoint of BU.	2) Definition of perpendicular bisector.
3) $\overline{BL} \cong \overline{LU}$	3) Definition of midpoint.
4) $\overline{EL} \cong \overline{EL}$	4) Reflexive property
5) $\angle BLE \cong \angle ULE$	5) Definition of perpendicular bisector.
6) $\triangle BEL \cong \triangle UEL$	6) LL Theorem

**LEARNING ACTIVITY 11.4.2.1 to 11.4.2.2**

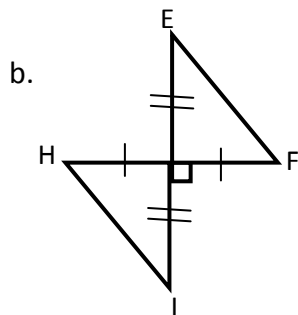
20 minutes

1. Identify whether the given pairs of triangles are congruent or not. State the postulate or theorem that proves that the pair of triangles are congruent.



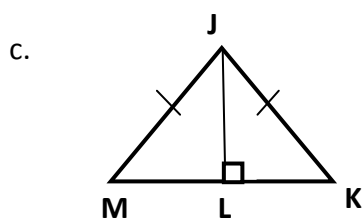
Statement: _____

Reason: _____



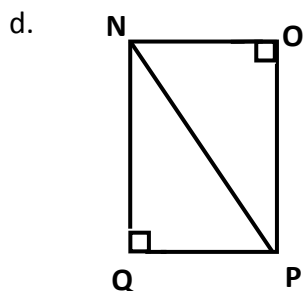
Statement: _____

Reason: _____



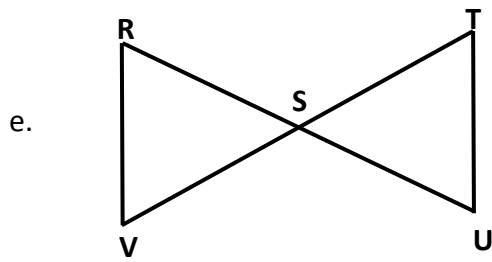
Statement: _____

Reason: _____



Statement: _____

Reason: _____



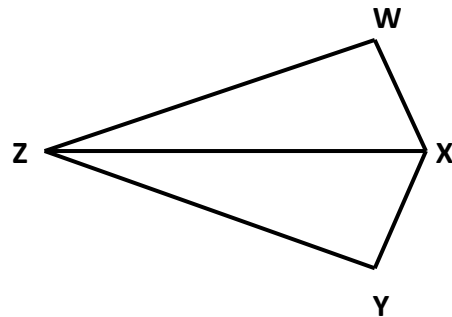
Statement: _____

Reason: _____

2. Complete the missing statement or reason below to prove that the triangles are congruent.

- a. Given: $\angle W$ and $\angle Y$ are right angles
 $\overline{ZX}$ bisects $\angle WXY$

Prove: $\overline{WX} \cong \overline{YX}$



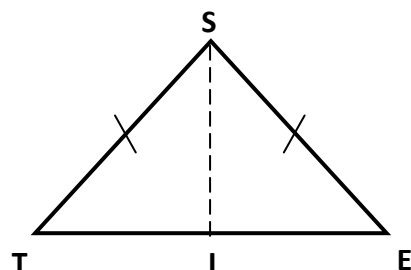
Proof:

Statements	Reasons
1) $\angle W$ and $\angle Y$ are right angles.	1) Given
2) $\overline{ZX}$ bisects $\angle WXY$.	2) _____
3) $\angle WXZ \cong \angle YXZ$	3) _____
4) $\overline{ZX} \cong \overline{ZX}$	4) _____
5) $\triangle WXZ \cong \triangle YXZ$	5) _____
6) $\overline{WX} \cong \overline{YX}$	6) _____

3. Write a two-way proof.

Given: $\triangle SET$ is an isosceles triangle with
 $\overline{ST} \cong \overline{SE}$
 $\overline{SI}$ bisects $\angle S$.

Prove: $\angle T \cong \angle E$

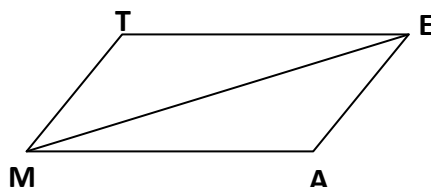




11.4.2.3 Solving Problems on Congruent Triangles

Problem 1

A diagonal EM is drawn from parallelogram TEAM. Prove that the two triangles formed by the diagonal are congruent.



Since the given figure is a parallelogram, we write the given based from the properties of a parallelogram.

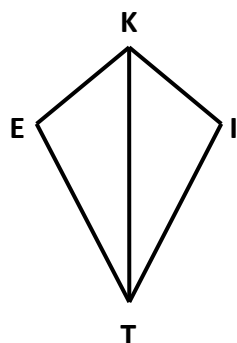
Solution

$\overline{TE} \cong \overline{MA}$; $\overline{TM} \cong \overline{EA}$	A rectangle has two pairs of opposite and congruent sides.
$\angle T \cong \angle A$	Opposite angles are congruent.
$\overline{EM} \cong \overline{EM}$	Reflexive Property
$\triangle TEM \cong \triangle EAM$	
Proof:	AAS or SAA Theorem

Problem 2

A kite maker, Mark, makes congruent triangles by drawing a diagonal. How can he be sure that the triangles are congruent? What postulate will guarantee that the triangles are congruent?

Solution



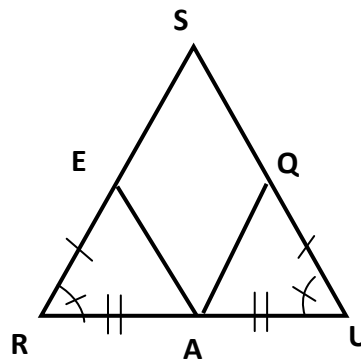
$\angle E \cong \angle I$	A kite has one pair of opposite angles that are congruent.
$\angle EKT \cong \angle IKT$	The diagonals of a kite bisect opposite angles.
$\overline{KT} \cong \overline{KT}$	Reflexive Property
Proof:	AAS or SAA Theorem



Problem 3

Given: $\overline{SR} \cong \overline{SU}$
 $\overline{RE} \cong \overline{UQ}$
A is the midpoint of $\overline{RU}$

Prove: $\overline{AE} \cong \overline{AQ}$



Plan for proof:

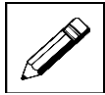
By applying the isosceles triangle theorem using the base angles, $\angle R \cong \angle U$. The markings on the figure suggest that triangles AER and AQU may be proved congruent by using the SAS Postulate.

Proof:

Statements	Reasons
1) $\overline{SR} \cong \overline{SU}$	1) Given
2) $\angle R \cong \angle U$	2) Isosceles Triangle Theorem 1
3) A is the midpoint of $\overline{RU}$	3) Given
4) $\overline{RA} \cong \overline{AU}$	4) Definition of midpoint.
5) $\triangle AER \cong \triangle AQU$	5) SAS Postulate
6) $\overline{AE} \cong \overline{AQ}$	6) CPCTC

After studying properties of similar triangles that follow, answer this question.

Can similar triangles be congruent or congruent triangles be similar?

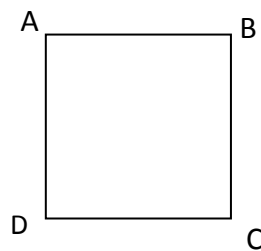
**LEARNING ACTIVITY 11.4.2.3**

20 minutes

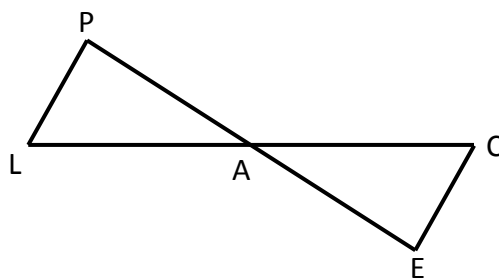
Solve the following problems involving congruent triangles.

1. You are tasked to make a design of the flooring on your room using triangles. The only materials available are square tiles. How are you going to make the design given that $\overline{AB} \cong \overline{BC}$ and $\overline{CD} \cong \overline{DA}$.

Show a proof using triangle congruence.

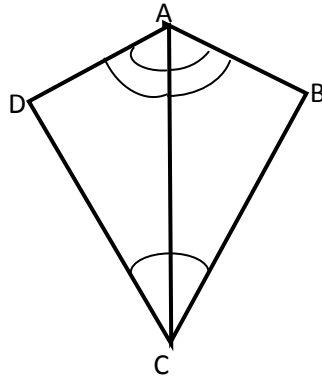


2. The figure below shows a way to find the distance from point P to point E (and point L to point C). If point A is the midpoint of $\overline{PE}$ and $\angle P \cong \angle E$, prove that $\triangle LPA \cong \triangle CEA$.

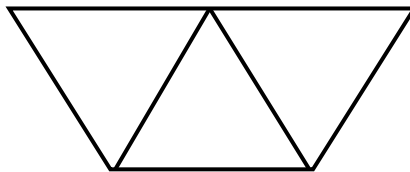




3. Show that ABCD is a kite. Use Proof of congruence.



4. The figure below shows an isosceles trapezium ACDE. If point B is the midpoint of $\overline{AC}$ and $\angle ABE \cong \angle CBD$, prove that $\triangle ABE \cong \triangle CBD$.





11.4.2.4 Similar Triangles

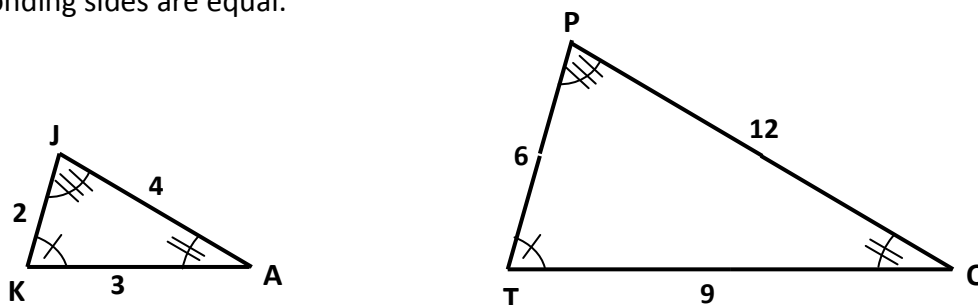
To prove that two triangles are similar, we have to show that the triangles satisfy the definition of similar polygons. The word similar expressed in symbol as $\sim$ involves corresponding angles and corresponding sides. When we say two triangles are similar, we may conclude that all the corresponding angles are congruent and that all the corresponding sides are proportional.

The concept of ratio and proportion will be useful in proving similar triangles. It is necessary to state a postulate about similar triangles to shorten proofs and provide us with a more direct method.

Similar Triangles

Two triangles are similar if and only if their vertices can be made to correspond in such a way that corresponding angles are congruent and the corresponding sides are proportional.

Corresponding sides of similar triangles are proportional means that the ratio of the lengths of corresponding sides are equal.



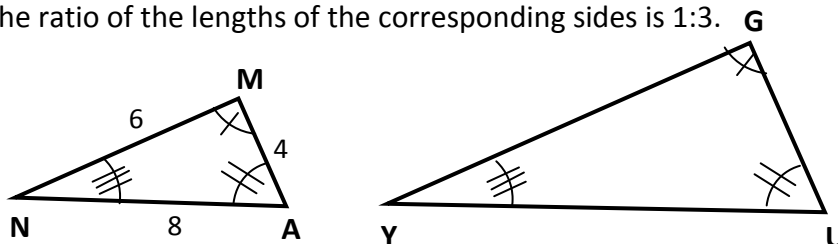
For similar triangles JAK and POT, we have:

Three pairs of congruent angles $\angle K \cong \angle T$, $\angle A \cong \angle O$ and $\angle J \cong \angle P$
 Three pairs of corresponding sides which are proportional

$$\frac{JK}{PT} = \frac{KA}{TO} = \frac{AJ}{OP} \quad \frac{2}{6} = \frac{3}{9} = \frac{4}{12} \quad \frac{1}{3} = \frac{1}{3} = \frac{1}{3}$$

The common ratio of the corresponding sides is called the **ratio of similarity**. The ratio of similarity is 1:3.

Example If $\triangle MAN \sim \triangle GUY$, $MA = 4$, $AN = 8$ and $NM = 6$ units, find all the sides of $\triangle GUY$ if the ratio of the lengths of the corresponding sides is 1:3.





Solution

$$\frac{MA}{GU} = \frac{AN}{UY} = \frac{NM}{YG}$$

$$\frac{4}{GU} = \frac{8}{UY} = \frac{6}{YG}$$

$$\frac{1}{3} = \frac{1}{3} = \frac{1}{3}$$

$$\frac{4}{GU} = \frac{1}{3}$$

$$\frac{8}{UY} = \frac{1}{3}$$

$$\frac{6}{YG} = \frac{1}{3}$$

$$GU = 12$$

$$UY = 24$$

$$YG = 18$$

Alternatively, if the ratio is 1:3, then enlargement factor $k = \frac{3}{1} = 3$. And $\mathbf{B} = k\mathbf{A}$, where B is the second triangle and A is the first triangle, we write equations as the following and solve.

$$GU = kMA$$

$$GU = 3MA$$

$$GU = 3 \times 4 = 12 \text{ units}$$

$$UY = kNA$$

$$UY = 3NA$$

$$UY = 3 \times 8 = 24 \text{ units}$$

$$YG = kNM$$

$$YG = 3NM$$

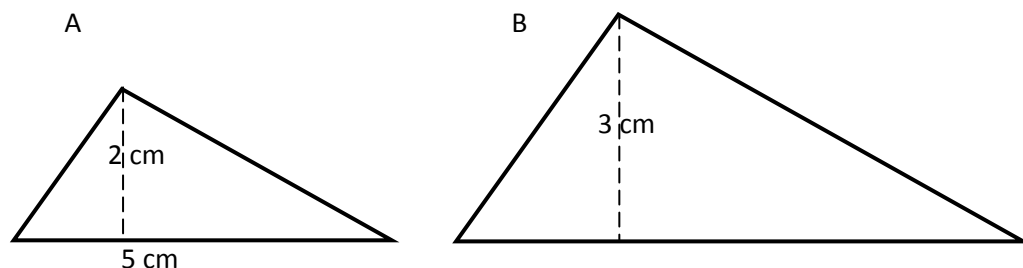
$$YG = 3 \times 6 = 18 \text{ units}$$

Therefore the sides are GU = 12units, UY = 24units and YG = 18units.

The equation is always expressed as $\mathbf{B} = k\mathbf{A}$. If the ratio of similarity is known, k value is known or can easily be worked out.

And k is less than 1 ($k < 1$) when the subsequent shape (B) is smaller, then a reduction factor is found. But k is greater than 1 ($k > 1$) as in our example since the sides of the subsequent shape are larger.

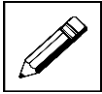
Let look at how we can find k from a problem, when the ratio of similarity is not stated but the information is given.



Since the corresponding heights are given, we find the k-factor by dividing the heights. Note that the triangle B is larger than triangle A.

$$k = \frac{H}{h} = \frac{3}{2} \text{ or } 1.5 \text{ Enlargement factor, hence } B = 1.5A$$

$$k = \frac{h}{H} = \frac{2}{3} \text{ Reduction factor, hence } A = \frac{2}{3}B$$

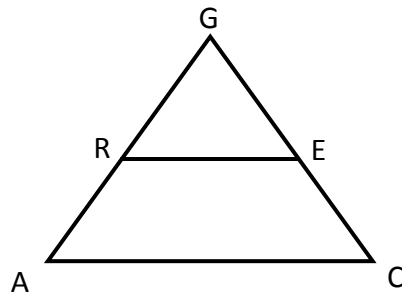


LEARNING ACTIVITY 11.4.2.4



10 minutes

Complete each proportion using the figure of similar triangles below.



1. $\frac{\overline{GR}}{\overline{RA}} = \frac{\overline{GE}}{?}$

6. $\frac{\overline{GE}}{\overline{ER}} = \frac{?}{\overline{CA}}$

2. $\frac{\overline{GA}}{?} = \frac{\overline{GC}}{\overline{GE}}$

7. $\frac{\overline{GE}}{\overline{GC}} = \frac{\overline{GR}}{?}$

3. $\frac{\overline{GR}}{\overline{RE}} = \frac{?}{\overline{AC}}$

8. $\frac{?}{\overline{GE}} = \frac{\overline{GA}}{\overline{GC}}$

4. $\frac{?}{\overline{CA}} = \frac{\overline{GE}}{\overline{ER}}$

9. $\frac{\overline{RE}}{\overline{EG}} = \frac{?}{\overline{CG}}$

5. $\frac{\overline{CA}}{\overline{CG}} = \frac{\overline{ER}}{?}$

10. $\frac{\overline{CA}}{\overline{AG}} = \frac{\overline{ER}}{?}$

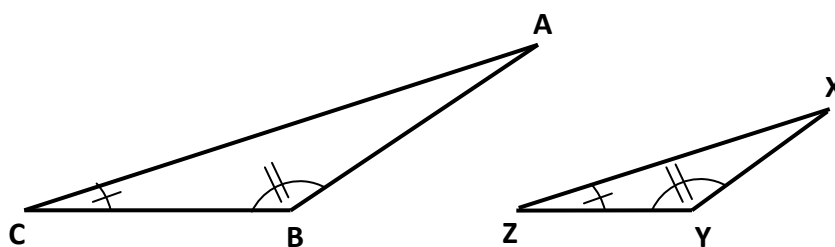


11.4.2.5 Proving the Similarity of Triangles

Similarity between triangles is easier to prove as well as triangle congruence because the shape of a triangle is completely determined by the measures of its angles. We will use this postulate in problems that involve similar triangles.

AA Postulate

If two angles of a triangle are congruent to two angles of another triangle, then the two triangles are similar.



According to the AA Postulate:

If $\angle C \cong \angle Z$ and $\angle B \cong \angle Y$, then $\triangle ABC \sim \triangle XYZ$.

If we wish to prove that two triangles are similar, we do not need to depend on the definition of similar polygons. The AA Postulate provides us with an efficient method.

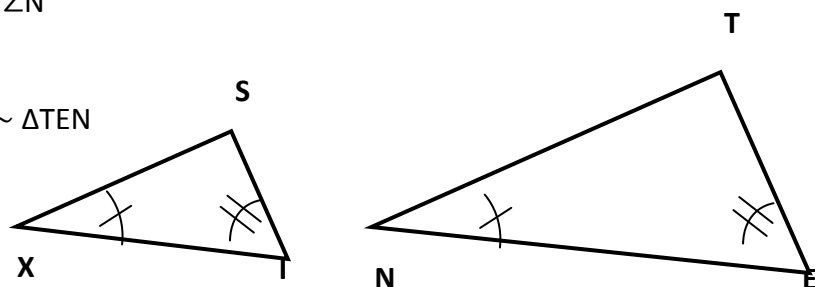
If we are given two triangles and that the corresponding two pairs of angles are congruent, we can conclude that the corresponding angles of the third pair are also congruent and therefore the two triangles are similar by the AAA Theorem.

AAA Similarity Theorem

If the corresponding angles of two triangles are congruent, then the two triangles are similar.

Example Given: $\angle X \cong \angle N$
 $\angle I \cong \angle E$

Prove: $\triangle SIX \sim \triangle TEN$



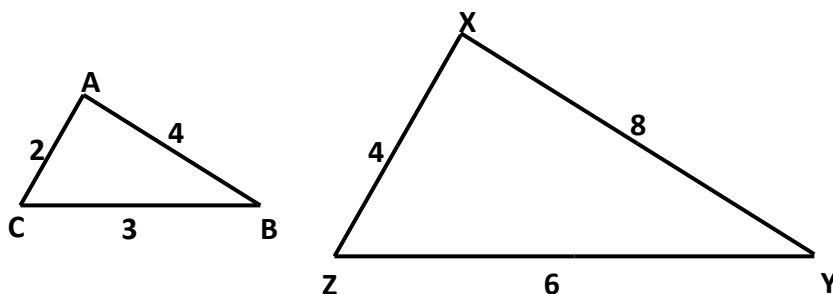


Proof:

Statements	Reasons
1) $\angle X \cong \angle N$ $\angle I \cong \angle E$	1) Given
2) $m\angle X = m\angle N$ $m\angle I = m\angle E$	2) Definition of congruent angles
3) $m\angle X + m\angle I = m\angle N + m\angle E$	3) Addition Property of Equality
4) $m\angle X + m\angle N + m\angle S = 180^\circ$ $m\angle N + m\angle E + m\angle T = 180^\circ$	4) The sum of the measures of the three angles in a triangle is equal to 180° .
5) $m\angle X + m\angle N + m\angle S = m\angle N + m\angle E + m\angle T$	5) Transitive Property of Equality
6) $m\angle S = m\angle T$	6) Subtraction Property of Equality
7) $\angle S = \angle T$	7) Definition of congruent angles
8) $\triangle SIX \sim \triangle TEN$	8) AAA Similarity Theorem

SSS Similarity Theorem

If the lengths of the three sides of one triangle are proportional to the corresponding three sides of another triangle, then the two triangles are similar.



Example: Find the ratios of the corresponding sides to prove that $\triangle ABC \sim \triangle XYZ$.

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX} \quad \frac{2}{4} = \frac{3}{6} = \frac{4}{8} \quad \frac{1}{2} = \frac{1}{2} = \frac{1}{2}$$

Since the corresponding sides are proportional, therefore $\triangle ABC \sim \triangle XYZ$ by SSS Similarity Theorem.

We use the next theorem as another way to show that two triangles are similar.

SAS Similarity Theorem

If the lengths of the two pairs of corresponding sides of two triangles are proportional and their respective included angles are congruent, then the two triangles are similar.

Theorems on SSS and SAS similarities may be associated to SSS and SAS Postulates on congruent triangles. However, the difference between the Similarity Theorems and Congruence Postulates is that in proving triangle congruence, corresponding sides must be congruent while in proving the similarity of triangles, corresponding sides must be proportional.

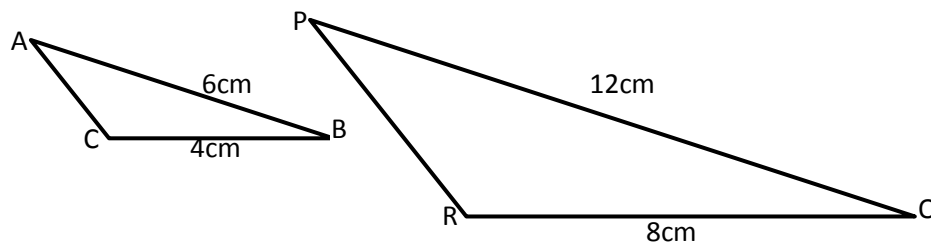
**Enlargement or Reduction Factor k (1-D)**

Sides of similar shapes increase or decrease by a factor k.

The factor $k < 1$ if it is a reduction factor such as $k = 0.8$ or $k = \frac{3}{4}$. The factor $k > 1$ if it is an enlargement factor such as $k = 3$, $k = 2.6$ or $k = \frac{8}{3}$.

Given a diagram of a plane, and the enlargement or reduction factor k, we can draw the similar figure by increasing or decreasing the corresponding sides by multiplying them by k factor.

Example Show that the two triangles are similar when AB corresponds to PQ, BC corresponds to QR and AC corresponds to PR.



Solution

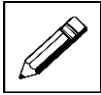
$$\text{enlargement factor} = \frac{PQ}{AB} = \frac{12}{6} = 2$$

$$\text{enlargement factor} = \frac{QR}{BC} = \frac{8}{4} = 2$$

Since both correspondences increased by 2 therefore the two triangles are similar.

A similar application can be made with 2-D and 3-D problems. For 2-D the area problem, we use k^2 and for 3-D the volume problem we use k^3 .

Thus $B = k^2A$ or $B = k^3A$ for 2-D (area) and 3-D (volume) respectively.



LEARNING ACTIVITY 11.4.2.5



10 minutes

Answer the following completely using similar triangles.

1. If $\triangle PIN \sim \triangle TAG$, then $\frac{\overline{PI}}{\overline{TA}} = \frac{\overline{IN}}{?}$.

2. If $\triangle MIS \sim \triangle TAH$, then $\frac{\overline{MI}}{\overline{IS}} = \frac{?}{\overline{AH}}$.

3. If $\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}}$, is $\overline{AB} \bullet \overline{EF} = \overline{DE} \bullet \overline{BC}$? Explain.

4. If $\triangle PHI \sim \triangle USA$, identify whether the following proportion is correct or not. Write a check mark (✓) if it is a proportion and a cross mark (X) if it is not.

a. $\overline{PH} \bullet \overline{US} = \overline{HI} \bullet \overline{SA}$

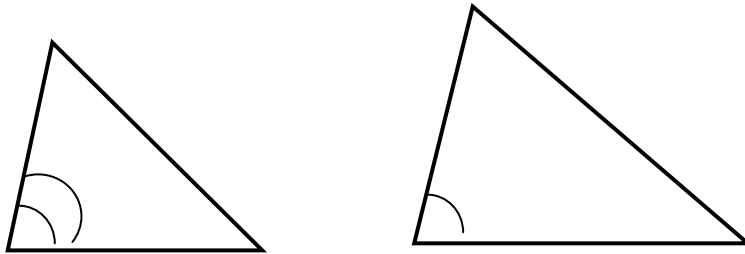
b. $\overline{PI} : \overline{HI} = \overline{UA} : \overline{SA}$

c. $\overline{PU} : \overline{SA} = \overline{HA} : \overline{US}$

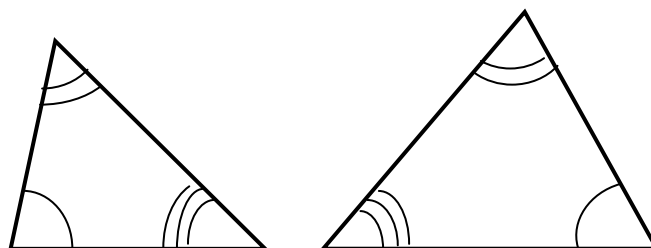
d. $\overline{UP} : \overline{HA} = \overline{PS} : \overline{IA}$



5. If $\triangle ABC \sim \triangle JKL$, identify whether the following proportion is correct or not. Write a check mark (✓) if it is a proportion and a cross mark (X) if it is not.



6. If a bases of triangles are in the ratio of 2 : 5, what is the
- a) Reduction factor k ?
 - b) Enlargement factor k ?
7. Evaluate value of B when $B = kA$ and $k = 2.5$ and $A = 4.2\text{cm}$.
8. Evaluate value of B when $B = kA$ and $k = 0.5$ and $A = 12\text{cm}$.
9. Evaluate value of k when $B = kA$ and $A = 20\text{cm}$ and $B = 70\text{cm}$.
10. If $\triangle ABC \sim \triangle LMN$, identify whether the following proportion is correct or not. Write a check mark (✓) if it is a proportion and a cross mark (X) if it is not.





11.4.2.6 Applications of Similar Triangles

Similar triangles are used to determine the height of tall objects using shadow lengths. How will you know the height of the tallest flagpole or a tower in the country without even actually climbing and or measuring it using any measuring instrument? The application of similar triangles by using a shadow in this case can be used. The only limitation is that this can only be applied in right triangles.

We define the concept of similar triangles as $\triangle ABC \sim \triangle A'B'C'$ whose sides and angles could be put into correspondence such that the following properties are true:

Property (i) : $A = A'$, $B = B'$ and $C = C'$

Property (ii) : $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

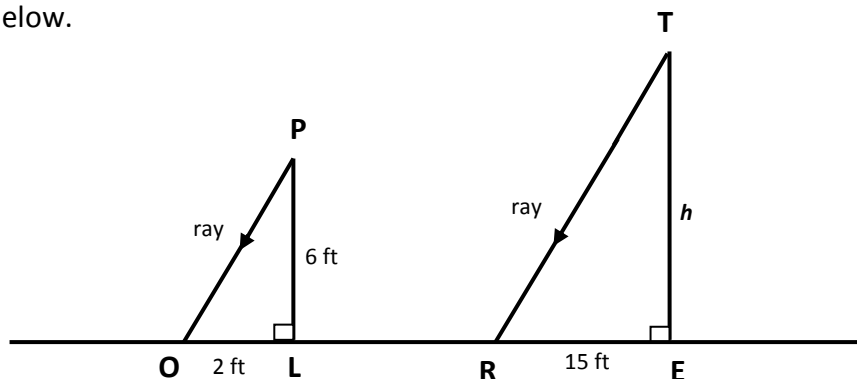
If property (i) is true, property (ii) is guaranteed to be true. Likewise, if property (ii) is true, then property (i) is also guaranteed to be true. Property (ii) may be derived using the proportion principle which is expressed as the product of the means equals the product of the extremes. Thus if $a:a' = b:b'$, then $ab' = a'b$.

We may use property (ii) conditions in order to calculate the unknown lengths in the triangles. Let us use these properties in solving problems involving similar triangles.

Example 1 A tree casts a 15-ft shadow while at the same time a pole 6 ft high casts a shadow of 2 ft. If both tree and pole make right angles with the ground, find the height (h) of the tree. (use metres)

Solution:

Shadows are formed for both of these objects (the tree and the pole) because the sun shines down on both objects and the rays of the sun strike the ground and form equal angles. This results in a pair of similar triangles being formed as shown in the figures below.





By comparing the lengths of the two shadows, against the two heights, using similar triangles, we can work out the unknown height of the tree.

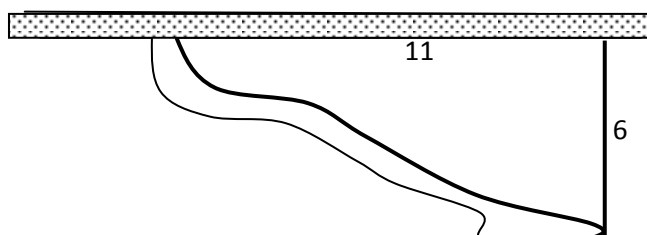
Since the rays of the sun form equal angles with the ground for both objects, therefore we can say that $\angle O \cong \angle R$. At the same time, the pole and the tree form right angles with the ground, hence $\angle L \cong \angle E$. Notice that the two angles are congruent and since we are asked to find the height of the tree, we may use AAS or SAA Theorem. Hence $\triangle POL \sim \triangle TRE$, and $h = 45$ ft.

**LEARNING ACTIVITY 11.4.2.6**

20 minutes

PROBLEM SOLVING

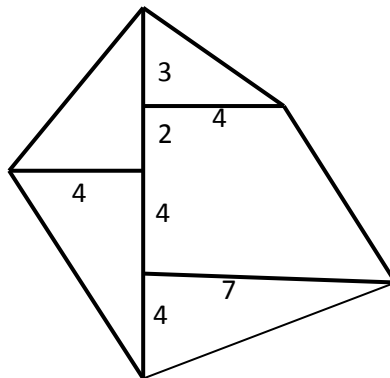
1. At some time in the afternoon, a 6-foot person casts a shadow that is 10 feet long while a nearby flagpole casts a shadow that is 22 feet long. To the nearest foot, how high is the flagpole?
2. A 30-ft ladder leans against the side of a building and reaches the flat roof. If the ladder stands at a point 10 feet from the house, how high is the house to the nearest foot.
3. Map scale was 1:10 000. If a triangular piece of land had dimensions of 3 cm and 7 cm on paper, calculate the actual land area.
4. Find actual land area for the map below. Scale 1 cm: 100 m





5. An area has base of 600 m and perpendicular height of 400 m. If it is to be drawn on paper, what will be the dimensions using a scale of 1:10 000?

6. Calculate total area of the off-set diagram below.



**SUMMATIVE TASK 11.4.2**

30 minutes

A. Multiple Choice: Write the letter of your choice before each number.

_____ 1. Which of the following is not a method for proving triangle congruence?

- a. ASA b. AAS c. SAS d. SSA

_____ 2. Two right triangles with the same perimeter are

- a. Always congruent c. sometimes congruent
b. Always equal in area d. never congruent

_____ 3. The supplement of every acute angle is

- a. Acute angle c. right angle
b. Obtuse angle d. reflex angle

_____ 4. In $\triangle XYZ$, $XY = 8$ and $YZ = 10$. Which of the following must be true?

- a. $XZ > 2$ b. $XZ < 2$ c. $XZ > 10$ d. $XZ < 10$

_____ 5. If $\triangle PIT \cong \triangle BUL$, then which of the following must be true?

- a. $\overline{PT} \cong \overline{BL}$ c. $\angle I \cong \angle L$
b. $\overline{IT} \cong \overline{BU}$ d. $\angle P \cong \angle U$

_____ 6. Which of the following cannot be used as a reason in a proof?

- a. definition c. yesterday's theorem
b. Postulate d. tomorrow's theorem

_____ 7. Which of the following are statements that are accepted to be true and no further proof is needed.

- a. Definition c. Postulates
b. Corollary d. Theorems

_____ 8. Which of the following describe a sequence of true statements placed in logical order?

- a. facts c. proof
b. Hypothesis d. reasons



_____ 9. In $\triangle ONE$, $ON = OE$, $m\angle O = 46$ and $\overline{NS}$ is an altitude, what is $m\angle ENS$?

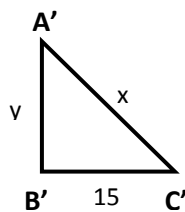
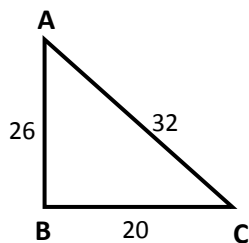
- a. 23 b. 44 c. 67 d. 134

_____ 10. If the corresponding angles of two triangles are congruent, then the two triangles are similar. This statement describe the

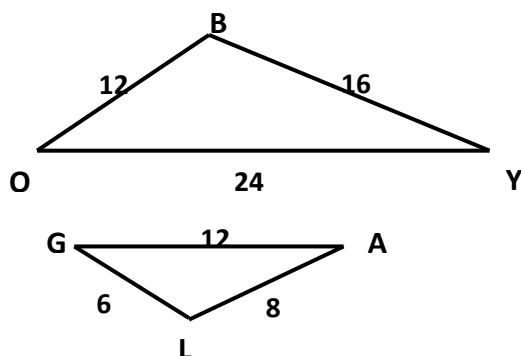
- a. AA Postulate c. AAS Congruence
b. AAA Similarity Theorem d. CPCTC

B. Problem Solving

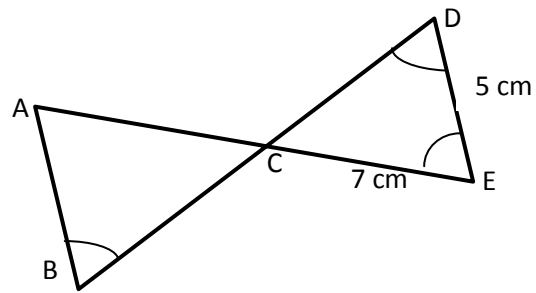
1. In similar triangles ABC and $A'B'C'$, find x and y if $\angle A \cong \angle A'$ and $\angle B \cong \angle B'$.



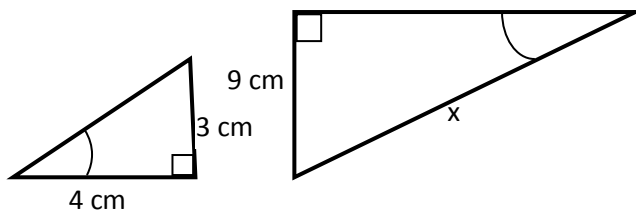
2. In each figure, determine the proportion needed to prove that the triangles are congruent.



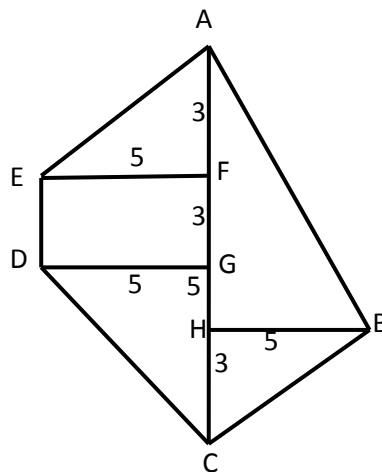
3. Given that the two triangles are congruent, find value of the side AB.



4. The two triangles are similar. Calculate value of side x .



5. Find area of the off-set diagram ABCDE. Measurements are in metres.



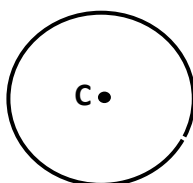


11.4.3: CIRCLES

Circles, just like triangles, are also found everywhere. We can see different circular figures like wheels, dishes, coins, watches and the like. Can you just imagine if the wheels used for transportation is triangular?

A **circle** is a connection of a set of all points equidistant from a fixed point called the **centre** while the fixed distance is the **radius**.

A circle is named by its centre and is denoted by the symbol $\odot$. Remember that the centre is not a point on the circle. The figure below can be named as circle C or $\odot C$.

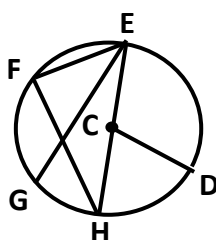


11.4.3.1 Parts of Circles

In the above definition of a circle, the radius is the distance from a point on the circle to the center. Another definition that can be used is as follows.

A **radius** is a segment drawn from a point of the circle on the circumference to its centre.

The plural of radius is **radii** and CD is just one of the radii of circle C. The other parts of the circle can be identified through the figure below.



$\overline{CE}$ and $\overline{CH}$ are also radii of circle C. Since $\overline{CE}$ and $\overline{CH}$ form a straight segment whose endpoints are on the circle, segment HE forms a diameter of the circle. Just like the radius, diameter $\overline{HE}$ is just one of the many diameters of circle O. You may notice that the diameter is formed by two radii whose endpoints are on the circle. This description will lead to the definition of a chord.

Points D, E, F, G and H are on the circumference of $\odot C$.



A **chord** is a line segment connecting any two points on the circle.

A **diameter** is a chord passing through the centre.

Based on the figure illustrated above, some of the chords of the circle C are $\overline{EF}$, $\overline{EG}$, $\overline{FH}$ and $\overline{HE}$ which happen to be a diameter also. If you try to draw some more chords on the circle, you will find that the diameter is the longest chord. The radius, chord and diameter of a circle are defined as segments. Some other parts of a circle are expressed through lines, rays or segments.

A **secant** is a line, ray or segment that contains a chord. It usually passes through two points on the circle.

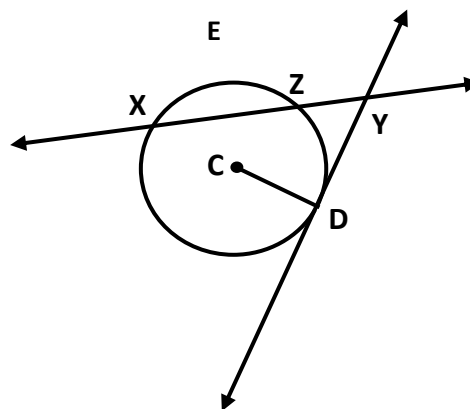
A **tangent** is a line, ray or a segment passing through exactly one point on the circle. The point of contact is called the **point of tangency**.

$\overline{XY}$ contains a chord ($\overline{XZ}$) on the circle.

Therefore $\overline{XY}$ is a secant of circle C.

$\overline{YD}$ is tangent to circle C and $\overline{OC}$ is tangent to $\overline{YD}$. Thus, we can say that $\overline{YD}$ and circle C are tangent and point D is the point of tangency.

From the chord $\overline{XZ}$, $\widehat{XEZ}$ is a minor arc containing minor segment XZE; while $\widehat{XDZ}$ is a major arc containing major segment XZD.



A **segment** is an area of a circle enclosed by an arc and a chord.

$\widehat{CZX}$ ($\angle XCZ < 180^\circ$) is a minor sector and $\widehat{CZX}$ ($\angle XCZ > 180^\circ$) is a major sector. In the diagram, combination of any two points on the circumference in X, E, Z and D and the centre forms a sector.

A **sector** is an area of a circle enclosed by two radii and an arc.

A sector contains central angle, which is an angle formed by two radii.

A chord and the radii form an isosceles triangle when the central angle is less than 90° . The height of the triangle can be found by using Pythagorean Formula when there is a need to calculate area of a segment.

Area of a segment is equal to area of sector minus area of the triangle formed the chord and the radii.



Name	Perimeter	Area
Circle	$C = \pi d$ or $C = 2\pi r$	$A = \pi \frac{d^2}{4}$ or $A = \pi r^2$
Sector	$P = 2r + l$	$A = \frac{\theta}{360} \times \pi r^2$
Segment	$P = c + l$	$A = \frac{\theta}{360} \times \pi r^2 - \frac{c}{2} \sqrt{r^2 - \frac{1}{4}c^2}$

In the area of a segment, c stands for the chord. You do not have to memorise, however based on geometry of circle the formula can be derived.

Example 1 Calculate area and perimeter of the circle given $d = 7$ cm. Use $\pi = 22/7$

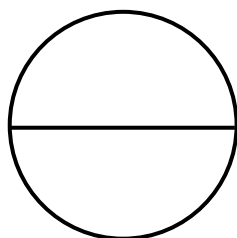
Solution

1. Area $A = \pi \frac{d^2}{4}$

$$A = \frac{22}{7} \times \frac{7^2}{4}$$

$$= \frac{11 \times 7}{2}$$

$$= 38 \frac{1}{2} \text{ cm}^2$$



2. Perimeter $C = \pi d$

$$C = \frac{22}{7} \times 7$$

$$= 22 \text{ cm}$$

Example 2 Calculate area and perimeter of the sector of a circle given that $d = 7$ cm and the central angle of the sector is 60° . Use $\pi = 22/7$

1. Area: since $d = 7$, $r = 3.5$

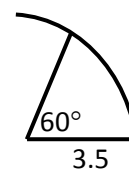
$$A = \frac{\theta}{360^\circ} \times \pi r^2 = \frac{60^\circ}{360^\circ} \times \frac{22}{7} \times 3.5 \times 3.5$$

$$= \frac{1}{6} \times 22 \times 0.5 \times 3.5$$

$$= \frac{1}{6} \times 38.5$$

$$= 6.416666...$$

$$= 6.42 \text{ cm}^2$$





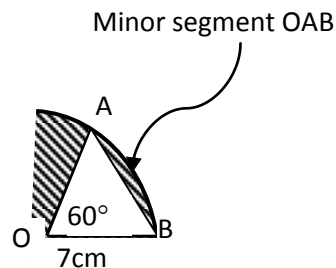
2. Perimeter $P = 2r + \ell$ where, $\ell = \frac{\theta}{360} \times C$

$$\begin{aligned} \ell &= \frac{60^\circ}{360} \times 22 & P &= 2 \times 3.5 + 3\frac{2}{3} \\ \ell &= 3\frac{2}{3} \text{ cm} & &= 10\frac{2}{3} \text{ cm} \end{aligned}$$

Example 3 Calculate area and perimeter of the minor segment OAB of a circle given that the chord $c = 7\text{cm}$ and $r = 7\text{cm}$ and the central angle of the sector is 60° . Use $\pi = \frac{22}{7}$

1. Segment Area (Minor) = Sector OAB – Triangle OAB

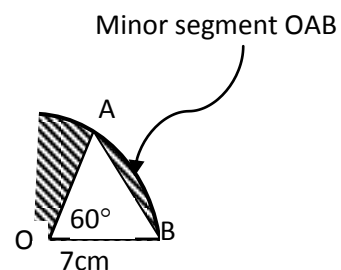
$$\begin{aligned} A &= \frac{\theta}{360} \times \pi r^2 - \frac{c}{2} \sqrt{r^2 - \frac{1}{4}c^2} \\ &= \frac{60^\circ}{360} \times \frac{22}{7} \times 7^2 - \frac{7}{2} \sqrt{7^2 - \frac{1}{4} \times 7^2} \\ &= \frac{1}{6} \times 22 \times 7 - \frac{7}{2} \sqrt{49 - \frac{49}{4}} \\ &= \frac{11}{2} - \frac{7}{2} \sqrt{\frac{196 - 49}{4}} \\ &= \frac{11}{2} - \frac{7}{2} \sqrt{\frac{147}{4}} \\ &= \frac{11}{2} - \frac{7}{2} \times \frac{\sqrt{3}}{2} \quad (\text{taken negative value so answer be positive}) \\ &= \frac{11}{2} + \frac{49}{4} \sqrt{3} \\ &= 5.5 + 21.2 \\ &= 26.6 \text{ cm}^2 \end{aligned}$$



2. Perimeter $P = c + \ell$ where, $\ell = \frac{\theta}{360} \times C$

$$\begin{aligned} \ell &= \frac{60^\circ}{360} \times \frac{22}{7} \times 7^2 & P &= 7 + 25\frac{2}{3} \\ & & &= 32\frac{2}{3} \text{ cm} \end{aligned}$$

$$\begin{aligned} \ell &= \frac{1}{6} \times 22 \times 7 \\ &= \frac{77}{3} \\ &= 25\frac{2}{3} \text{ cm} \end{aligned}$$



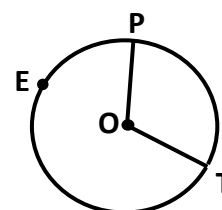


11.4.3.2 Angles in Circles

In statistics, circle graphs or pie charts, just like bar graphs and histograms, are very useful in presentation of data. In using a circle graph, each part is defined by an angle whose vertex is the center of the circle. This angle is called a central angle.

A **central angle** is an angle whose vertex is the centre of the circle

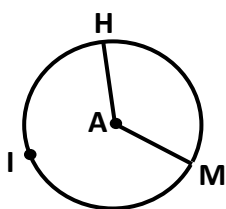
Based from the figure at the right, $\angle POT$ is a central angle and it may refer to the obtuse angle and/or the reflex angle. Notice that the central angle is formed by two radii. Likewise, a central angle may also be formed using diameters, secants or a combination of such.



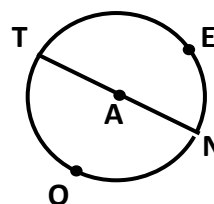
The central angle $\angle POT$ which forms an obtuse angle connects the endpoints of the two radii. This figure is not a segment, thus this is a **minor arc** which can be written as $\widehat{PT}$.

On the other hand, the reflex angle $\angle POT$ also forms an arc whose endpoints are points P and T but pass through point E. This is called a **major arc** and can be named as $\widehat{PET}$. These three points, P, E and T, were used in naming the major arc to avoid confusion. These arcs formed by central angles are called intercepted arcs.

Note that a minor arc is named by its endpoints while a major arc can be named using three points.



$\angle HAM$ is a central angle.
 $\widehat{HM}$ is a minor arc and
 $\widehat{HIM}$ is the major arc.



$\overline{TN}$ is a diameter.
 $\widehat{TEN}$ and $\widehat{HIM}$ are semicircles.

The **measure of a central angle** is proportional to the measure of the arc it intercepts. The **measure of a minor arc** is equal to the measure of its central angle. The **measure of a major arc** is equal to 360° - measure of the minor arc. The **measure of a semi circle** is 180° . Note that two circles being congruent in their radii are equal or congruent.



Example Given: $m\angle HAM = 120$ Find the measures of the minor and major arcs.

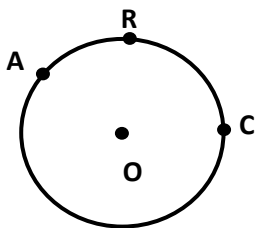
Solution

$$\begin{aligned}\widehat{HM} &= 120^\circ && \text{measure of the minor arc} \\ \widehat{HIM} &= 360^\circ - 120^\circ \\ \widehat{HIM} &= 240^\circ && \text{measure of major arc}\end{aligned}$$

Now that you know how to find the measure of minor and major arcs, let us now find the measure of central angles and arcs.

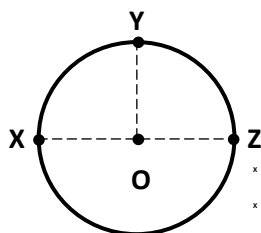
Arc Addition Postulate

The measure of the arc formed by two adjacent arcs is the sum of the measures of the two arcs. Adjacent arcs of a circle are arcs that share a common endpoint.



$\widehat{ARC}$ is formed by two adjacent arcs of circle O. By Arc Addition Postulate, if point R is the common endpoint of $\widehat{AR}$ and $\widehat{RC}$, then $m\widehat{ARC} = m\widehat{AR} + m\widehat{RC}$

The Arc Addition Postulate is also applied in finding the midpoint of an arc which is at the same time the common endpoint of the two arcs. The midpoint of an arc is the point of the arc that divides the arc into two equal arcs.



Using Arc Addition Postulate, $m\widehat{XYZ} = m\widehat{XY} + m\widehat{YZ}$. If radius $\overline{OY}$ is perpendicular to diameter $\overline{XZ}$, then point Y is the midpoint of $\widehat{XZ}$.

If point Y is the midpoint of $\widehat{XZ}$, then $m\widehat{XY} = m\widehat{YZ}$, thus $\widehat{XY}$ and $\widehat{YZ}$ are congruent arcs.

Congruent arcs are arcs in the same circle or in congruent circles that have equal measures.

This gives us the following theorems:

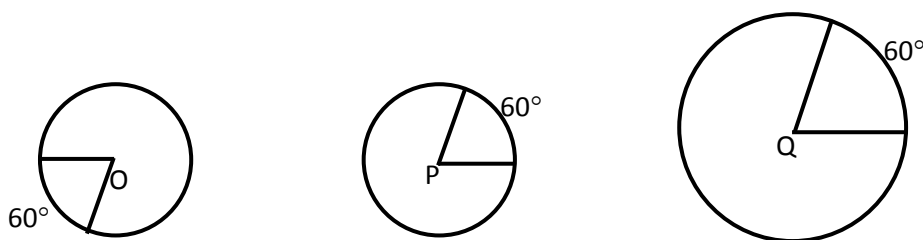
Central Angle-Arc Theorem

In the same circle or in congruent circles, two arcs are congruent if their central angles are congruent.

**Arc-Central Angle Theorem**

In the same circle or in congruent circles, two central angles are congruent if their corresponding minor arcs are congruent.

Remember that even if the arcs have the same degree measures, they are not congruent if they are contained in two different circles that are not congruent.

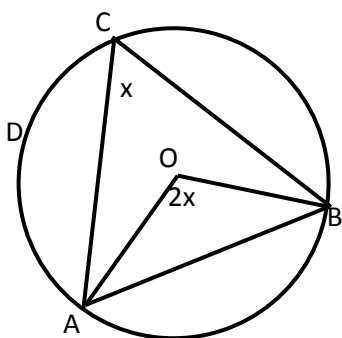


The circles O and P are congruent. Circle Q is similar to both circles O and P. The arc-central angle theorem may apply to circles O and P only.

Central Angle- Inscribed Angle

The angle subtended by an arc at the centre is twice the inscribed angle subtended by the same arc on the same segment.

Alternatively, we can say that the angle subtended by a chord at the circumference is half the angle at the centre subtended by the same chord.



Angles x and $2x$ are both on the same segment. The two angles are subtended by chord AB or arc AB.

Even if inscribed angle is made at point D, the angle is still in the same segment, therefore will measure half of that is made at the centre.

Angles on the same Segment

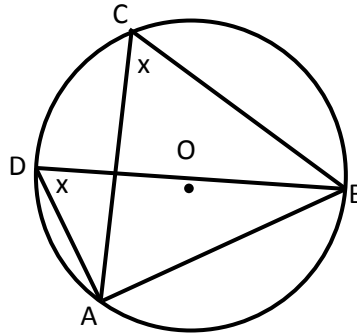
Inscribed Angles on the same segment subtended by the same arc or chord are equal.



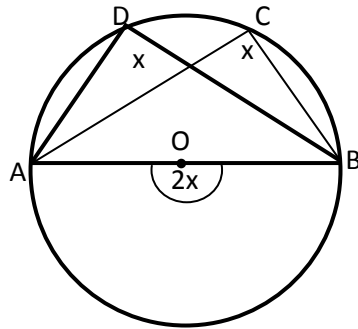
$$m\angle ADB = m\angle ACB$$

Both angles are on the same side of chord AB or are in the same segment.

Even if the two angles are formed in the minor segment, they would still be equal.



Based on the above two theorems, we can say that angles subtended by a diameter is 90° .



Since $\angle AOB$ is equal to 180° , $\angle ADB$ is equal to 90° .

The diameter contains the radii OA and OB, which contain the angle measure of $2x$.

**LEARNING ACTIVITY 11.4.3.1 to 11.4.3.2**

10 minutes

A. TRUE or FALSE: Write TRUE if the statement is correct otherwise write the word FALSE.

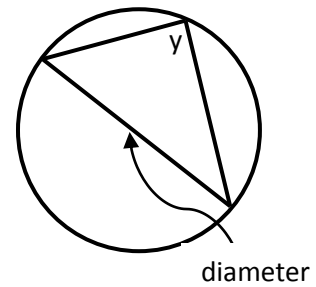
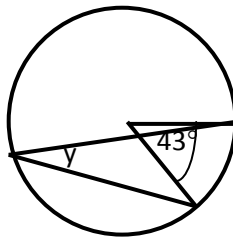
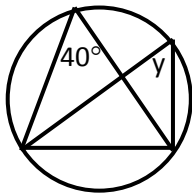
- _____ 1. A radius is a chord.
- _____ 2. A chord is a secant.
- _____ 3. A chord is a diameter.
- _____ 4. A diameter is a chord.
- _____ 5. A secant contains a chord.
- _____ 6. A central angle and its intercepted arc are congruent.
- _____ 7. A diameter determines two semicircles.
- _____ 8. The measure of a central angle is equal to the measure of its intercepted arc.
- _____ 9. If a line bisects a chord of a circle, then it bisects the minor arc of the chord.
- _____ 10. In the same circle or in congruent circles, two arcs are congruent if their central angles are congruent.
- _____ 11. If two tangents are parallel, then their point of tangency determines a diameter.
- _____ 12. The midpoint of an arc is the point of the arc that divides the arc into two equal angles.
- _____ 13. The intersection of all the diameters is the centre.
- _____ 14. Inscribed angle is an angle whose vertex is the centre of the circle.
- _____ 15. A tangent to a circle may contain the centre of the circle.
- _____ 16. A secant always contains points in the exterior of the circle.
- _____ 17. An angle whose vertex is the centre of the circle is an intercepted arc.
- _____ 18. The intersection of a secant and a circle is a chord.
- _____ 19. The radius is one-half the length of a chord.



_____ 20. The point inside the circle which is equidistant to any point on the circle is the centre.

B. Solve the following problems.

1. Find the values of angle y .



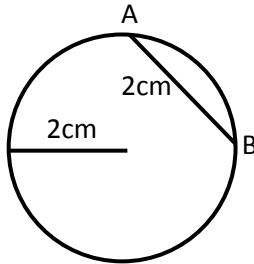
2. If a central angle is 60° is subtended by an arc of 3 cm, find the circumference.

3. A central angle is subtended by an arc of 5 cm. If the circumference of the circle is 30 cm, calculate the central angle.



4. A minor arc measures 20 cm is in the ratio of 1:4 to the major arc. Find the area of the sector contained in the major arc.

5. Calculate the central angle marked x subtended by 2 cm long chord AB in the circle below.

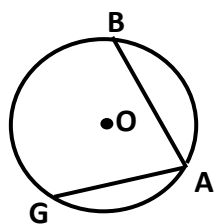




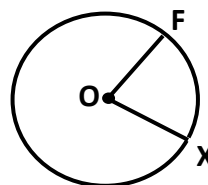
11.4.3.3 Inscribed Angles and Arcs

You learnt that central angles form minor arcs. An intercepted arc can also be formed by another angle which is inscribed in a circle.

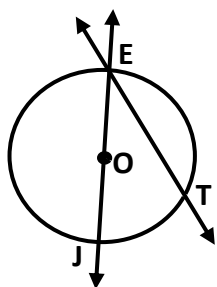
An **inscribed angle** is an angle whose vertex is on the circle and whose sides contain chords of the circle.



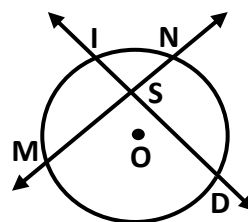
$\angle BAG$ is an inscribed angle



$\angle FOX$ is a central angle



$\angle JET$ is an inscribed angle and there is no central angle.



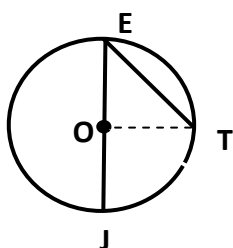
There is no inscribed angle nor central angle.

For any inscribed angle, the intercepted arc and the inscribed arc combine to form a circle. An intercepted arc is the part of a circle that corresponds to an angle and contains the arc between the sides of the angle including the endpoints while an inscribed angle is described by the theorem below.

Inscribed Angle Theorem

An inscribed angle is equal in measure to one half its intercepted arc.

In applying the theorem on inscribed angle, let us familiarize ourselves on the following example.



In circle O, one side of inscribed $\angle JET$ is a diameter $\overline{JE}$. $\angle JET$ is inscribed in $\widehat{JT}$.

To prove that $m\angle E = \frac{1}{2} (m \widehat{JT})$, we draw a radius OT .

Since all radii are congruent, $\overline{OE} \cong \overline{OT}$.



Therefore in $\triangle TOE$, $\angle T \cong \angle E$ (Isosceles triangle)

Revisiting the definition of exterior angle of a triangle that is equal to the sum of two remote interior angles, we say that $m\angle E + m\angle T = m\angle JOT$.

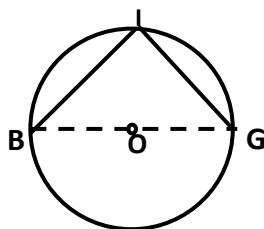
Because $\angle T \cong \angle E$, this equation becomes $m\angle E + m\angle E = m\angle JOT$ or $2m\angle E = m\angle JOT$.

If we divide both equation by 2, $m\angle E = \frac{1}{2} (m\angle JOT)$

Because the minor arc is equal to in measure to its central angle, that is, $m\angle JOT = m \text{ arc } JT$,
By substitution, we can prove that $m\angle E = \frac{1}{2} (m \widehat{JT})$.

Angle diameter theorem

If an inscribed angle intercepts a semicircle, then the angle is a right angle.



$$\begin{aligned} m\angle BIG &= 90^\circ \\ m\angle BIG &= \frac{1}{2} (m \text{ arc } BG) \\ m\angle BIG &= \frac{1}{2} (180^\circ) \\ m\angle BIG &= 90^\circ \end{aligned}$$

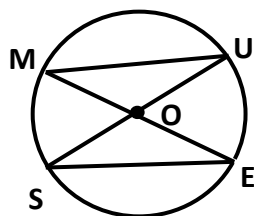
By the inscribed angle theorem, the measure of an inscribed angle in a circle is equal in measure to half of the arc it intercepts. The theorems that involve arc measure are shown below.

Congruent Arc Theorem

In the same or congruent circles, inscribed angles with congruent intercepted arcs are congruent.

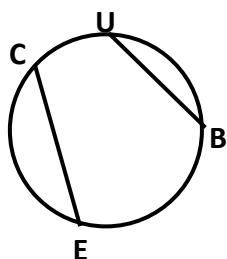
Congruent Angle Theorem

In the same or congruent circles, congruent inscribed angles have congruent intercepted arcs.



In circle O, inscribed angles MUS and MES both intercept arc MS. Thus, $\angle MUS$ and $\angle MES$ are congruent.

We know that a minor arc can either be formed through a central angle or an inscribed angle. In the succeeding theorems, you will learn that a minor arc can be formed not by an angle but by a chord. Likewise, angles in a circle formed by chords are not always inscribed angles. Let us explore the relationships among arcs, chords and diameters.

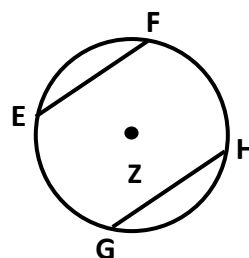
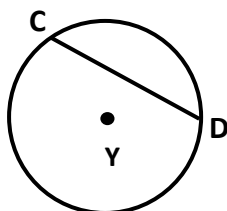
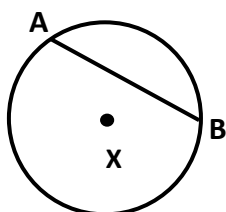


In the figure, chord UB and arc UB have the same endpoints, likewise for chord CE and arc CE . Both $\widehat{UB}$ and $\widehat{CE}$ are minor arcs.

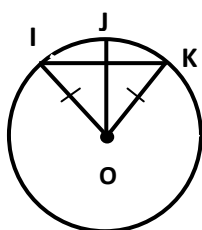
Arc-Chord Theorem

In the same circle or congruent circles, congruent arcs have congruent chords.

Example



In the given figures, if circle X is congruent to circle Y and chord $\overline{AB} \cong \overline{CD}$, then arc $\widehat{AB}$ and arc $\widehat{CD}$ are also congruent. Likewise, if $\overline{EF} \cong \overline{GH}$ in circle Z , then arc $\widehat{EF} \cong \widehat{GH}$.



As shown in the figure, $\overline{OJ}$, $\overline{OI}$ and $\overline{OK}$ are all radii and therefore are congruent. If radius $\overline{OJ}$ bisects arc $\widehat{IK}$, then point J is the midpoint of arc $\widehat{IK}$ and arc $\widehat{IJ} \cong \widehat{JK}$. Likewise, $\overline{OJ}$ bisects chord $\overline{IK}$. Therefore, $\overline{OJ} \perp \overline{IK}$. The following theorems state the relationship about a radius (or a diameter) for being a perpendicular bisector.

Radius Bisects Chord Theorem

In a circle, if a radius bisects a chord and its arc, then the radius is perpendicular to the chord.

Diameter Perpendicular Bisector to Chord Theorem

If a diameter of a circle is perpendicular to a chord, then the diameter bisects the chord and the arc.

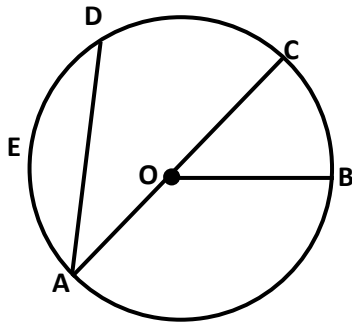
Chord Bisects Arc Theorem

If a chord of a circle bisects a second chord and its arc, then the first chord is a diameter and is perpendicular to the second chord.



Chords

If a circle has a centre O , then it is circle O . A **chord** is a straight line which joins two points on a circle. A **diameter** is a chord which runs through the centre of the circle.



Definition of Circle : In Circle O

O is the centre

AC is the diameter

OB is the radius

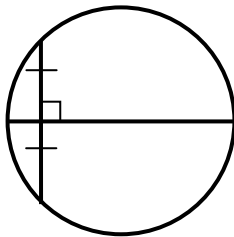
AD is a chord

AB, BC, CD, DE and EA are arcs

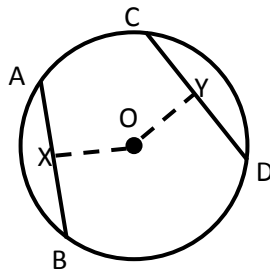
BOC and BOA are sectors

ADE is minor segment, ADB is major segment

Properties of Chords



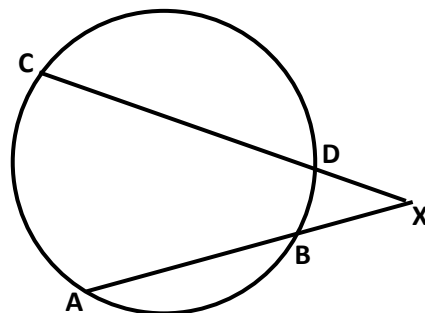
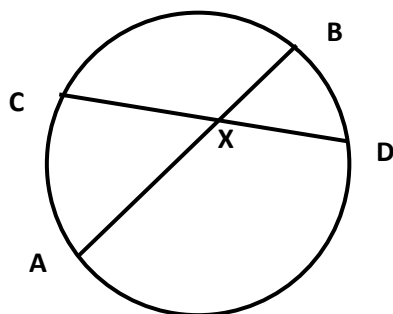
If a diameter is perpendicular to a chord then it bisects the chord.



If two chords are equal in length then they are equidistant from the centre of the circle.

The converse is also true: chords which are equidistant from the centre are equal in length.

In the figure above, if the centre of the circle is O and the centre of AB is X and centre of CD is Y then, $OX = OY$ and $AB = CD$.

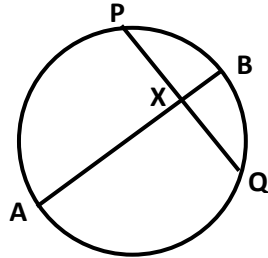


Given the two figures above $CX \cdot XD = AX \cdot XB$.



If two chords intersect within the circle or outside the circle, then the product of the line segments are equal.

Example 1 Chord PQ is 10 cm in length. What is the length PX if AB is a diameter, and is perpendicular to AB at X?

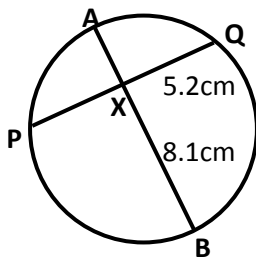


Solution

Since AB is perpendicular to PQ at X, PQ is bisected at X, and PQ = 10cm

$$\begin{aligned} PX &= \frac{1}{2} PQ \\ &= \frac{1}{2} \times 10 \text{ cm} \\ &= 5 \text{ cm} \end{aligned}$$

Example 2 What is the length **AX** if **AB** is a diameter, and is perpendicular to **AB**?

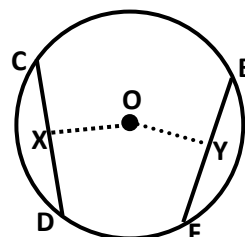


Solution

Since AB is perpendicular to **PQ** at X, **PQ** is bisected at X, and **XQ** = 5.2cm

$$\begin{aligned} PX \cdot XQ &= BX \cdot XA \\ 5.2 \times 5.2 &= 8.1 \times XA \\ 27.04 \div 8.1 &= XA \\ \mathbf{AX} &= 3.3\text{cm} \end{aligned}$$

Example 3 If **OX** = **OY**, X is midpoint of **CD**, Y is midpoint of **EF** and **CD** = 4.3 cm, what are lengths **EF** and **EY** ?





Solution

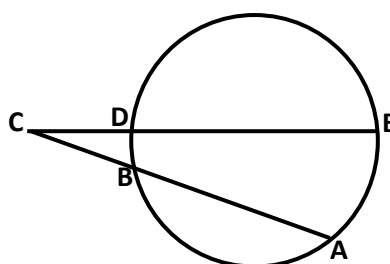
Since chord **CD** and chord **EF** are equidistant from centre O at X and Y respectively

$$\mathbf{CD} = \mathbf{EF} \quad (\text{since } \mathbf{OX} = \mathbf{OY} \text{ Given})$$

$$4.3\text{cm} = \mathbf{EF}$$

$$\mathbf{EF} = 4.3 \text{ cm}$$

Example 4 Secants **AC** and **EC** intersect at C. Given that segments **CD** = 2 cm, **DE** = 6.2 cm and **CB** = 2.5 cm, calculate the length of chord AB.



Solution

Given Chord **AB** and chord **ED** intersect at C.

$$\mathbf{AB} \cdot \mathbf{BC} = \mathbf{ED} \cdot \mathbf{DC}$$

$$\mathbf{AB} \cdot 2.5 = 6.2 \times 2$$

$$\mathbf{AB} \times 2.5 = 12.4$$

$$\mathbf{AB} = 12.4 \div 2.5$$

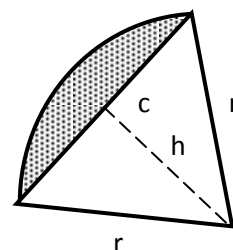
$$= \mathbf{4.96}$$

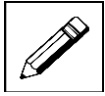
Therefore chord AB equals 4.96 cm

To find either **OD** or **OC** in example 3, use Pythagorean Theorem. Where $\mathbf{OD}^2 = \mathbf{DX}^2 + \mathbf{OX}^2$. And **CD** = 4.3cm therefore **DX** = 2.15cm. However, **OX** is not given so we cannot calculate **OD**. You can apply the theorem were applicable.

Say a triangle in a sector has sides r, r and c and the height h. To find area of triangle you need to calculate height h before finding the area. Or even to find area of minor segment, you need to know the area of the triangle in order to subtract it from area of sector. And h is always a perpendicular bisector. Thus

$$\begin{aligned} r^2 &= h^2 + \left(\frac{c}{2}\right)^2 & \text{where } h &= \sqrt{r^2 - \frac{c^2}{4}} \\ &= h^2 + \frac{c^2}{4} & A &= \frac{c}{2} \sqrt{r^2 - \frac{c^2}{4}} \end{aligned}$$

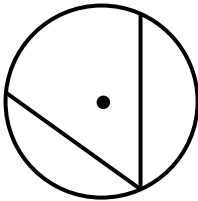


**LEARNING ACTIVITY 11.4.3.3**

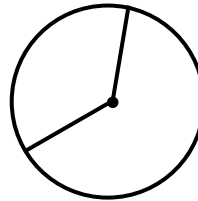
20 minutes

A. State whether the given angle is an inscribed angle or not by writing YES or NO.

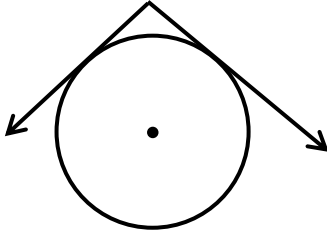
1.



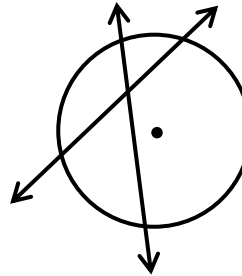
4.



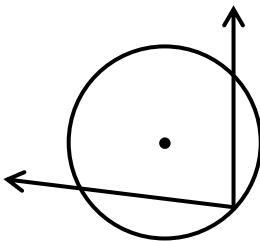
2.



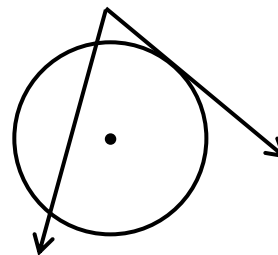
5.



3.

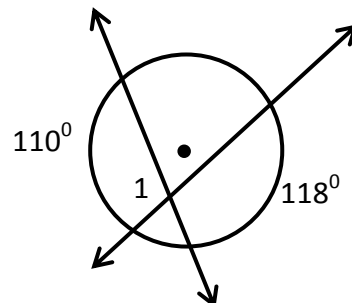


6.

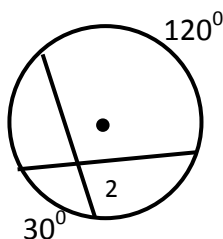


B. Find the measures of $\angle 1$ and $\angle 2$.

1.

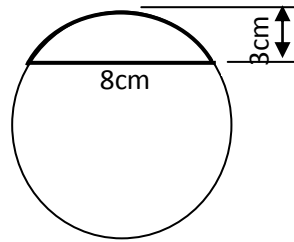


2.

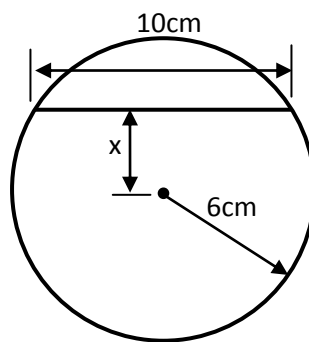




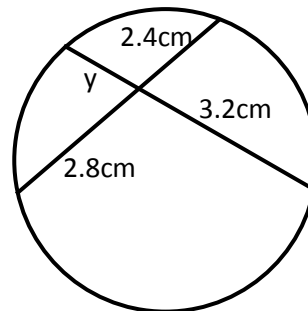
1. Calculate the diameter of the circle.



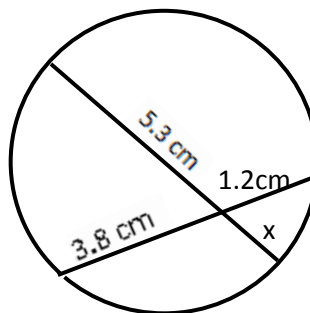
2. Find the distance x .



3. Find the length of y in the figure below.

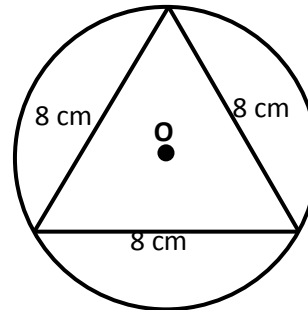


4. Find the length x .

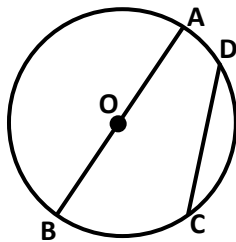




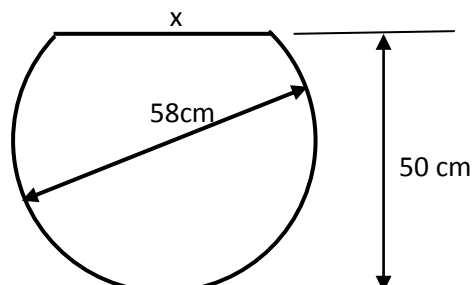
5. Figure below shows inscribed equilateral triangle. Calculate diameter of the circle.



6. The diameter **AB** is 30 cm long. If the chord **CD** is 9 cm from the centre, calculate the length of the chord CD.



7. Calculate the length of the chord x

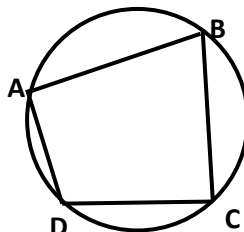




11.4.3.4 The Cyclic Quadrilaterals

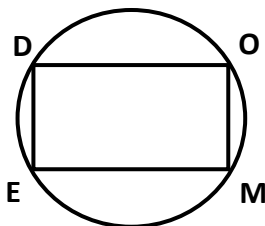
Previously or earlier, we learnt that an angle can be inscribed in a circle and in an inscribed triangle. In this case, a quadrilateral can also be inscribed in a circle as long as the vertices lie on the circle. A quadrilateral is said to be a cyclic quadrilateral if there is a circle passing through all of its four vertices.

$$\begin{aligned} m\angle A + m\angle C &= 180^\circ \\ m\angle B + m\angle D &= 180^\circ \end{aligned}$$



In cyclic quadrilateral, the opposite angles are supplementary.

Example 1 DOME is a parallelogram. Show that it is a rectangle.



Solution:

$$\angle D + \angle M = 180^\circ$$

$$\angle D \cong \angle M \text{ (Opposite angles of a parallelogram are } \cong \text{)}$$

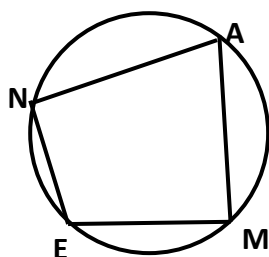
$$\angle D + \angle D = 180^\circ$$

$$2 \angle D = 180^\circ$$

$$\angle D = 90^\circ$$

Since $\angle D = 90^\circ$, then DOME is a rectangle.

Example 2 Quadrilateral NAME is inscribed in circle O. If $m\angle N = 96$ and $m\widehat{NAM} = 210$, find $m\angle A$, $m\angle M$, $m\angle E$, $m\widehat{NEM}$, and $m\widehat{AME}$.



Solution:

$$m\angle M = 180 - m\angle N$$

$$m\angle E = \frac{1}{2} m\widehat{NAM}$$

$$m\angle A = 180 - m\angle E$$

$$m\widehat{NEM} = 2 m\angle A$$

$$m\widehat{AME} = 2 m\angle N$$

$$m\angle M = 180 - 96$$

$$m\angle E = \frac{1}{2} (210)$$

$$m\angle A = 180 - 105$$

$$m\widehat{NEM} = 2 (75)$$

$$m\widehat{AME} = 2 (96)$$

$$m\angle M = 84$$

$$m\angle E = 105$$

$$m\angle A = 75$$

$$m\widehat{NEM} = 150$$

$$m\widehat{AME} = 192$$

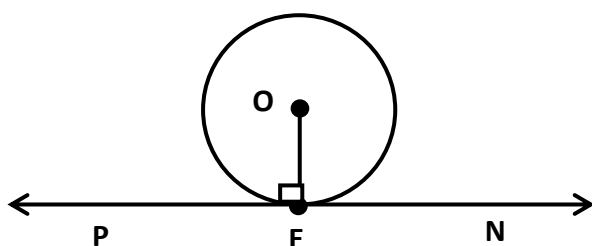


11.4.3.5 The Tangent to a Circle

Earlier we defined a **tangent** as a line, ray or a segment passing through exactly one point on the circle. The point of contact is called the **point of tangency**. This section will focus on properties of tangent lines and tangent circles.

Tangent Line Theorem

If a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of tangency.



In circle O, segment OE is a radius and appears to be perpendicular with $\overleftrightarrow{PN}$ to the point of tangency (point E).

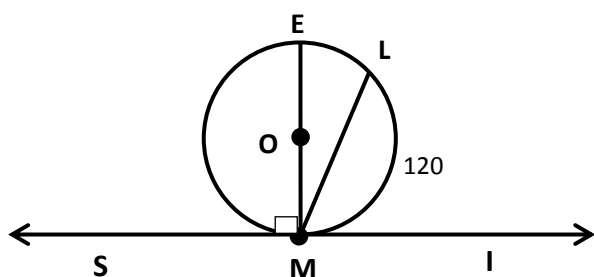
Likewise, this theorem may also be the same as the Tangent Line Theorem.

Tangent Line Theorem

If a line is perpendicular to a radius at its point on a circle, then the line is tangent to the circle.

Using the given figure, if $\overleftrightarrow{PN}$ is tangent to circle O at point E, then $m\angle POE = 90^\circ$.

The figure below shows the angle formed by a tangent and chords $\overline{ME}$ and $\overline{ML}$.



In circle O, $m\widehat{LM} = 120^\circ$ and $\overline{EM}$ is a diameter. What is the measure of $\angle SML$?

Solution:

$$m\angle SML = \frac{1}{2} m\widehat{MEL}$$

Since the circumference of a circle is 360° ,

$$360^\circ = m\widehat{MEL} + m\widehat{LM}$$

$$m\widehat{MEL} = 360^\circ - m\widehat{LM}$$

$$m\widehat{MEL} = 360^\circ - 120^\circ$$

$$m\widehat{MEL} = 240^\circ$$

$$m\angle SML = \frac{1}{2} (240^\circ)$$

$$m\angle SML = 120^\circ$$

Let us now study tangent circles.



Two circles are tangent circles when they are tangent to the same line at the same point of tangency. The circles are said to be **tangent internally** when they lie on the same side of the line and **tangent externally** when they lie on opposite sides of the line.

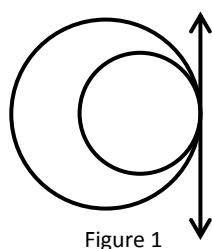


Figure 1

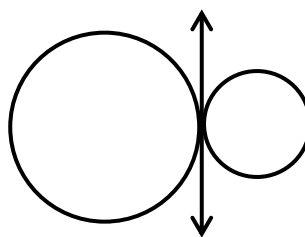
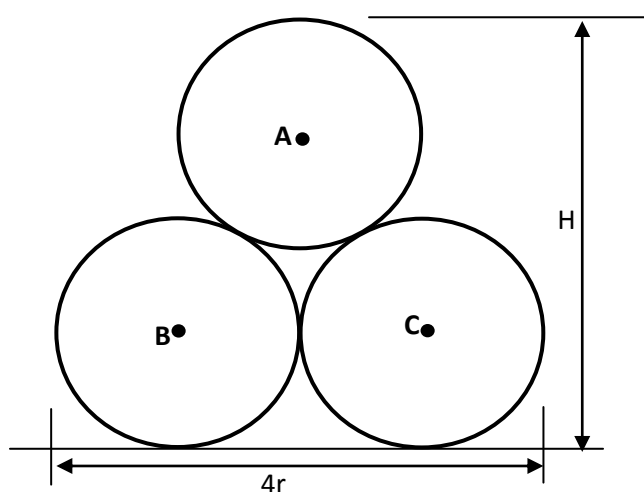


Figure 2

Circles in Figure 1 are tangent internally while circles in Figure 2 are tangent externally.

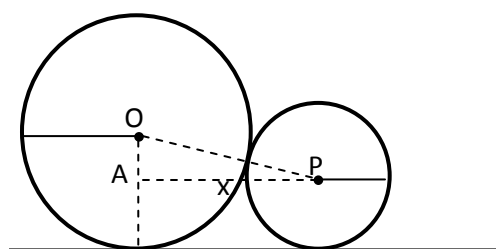


In Triangle ABC

$$P = 2r + 2r + 2r$$

$$A = \sqrt{3}r^2$$

The circles above are congruent, and each pair is tangent externally. Their centres form an equilateral triangle. Hence, each side of the equilateral triangle is $2r$ and the perpendicular height is $\sqrt{3}r$, which is obtained by Pythagoras theorem. The horizontal distance covered by two externally tangent circles is $4r$ and the vertical height of the above congruent circles can be given as $H = \sqrt{3}r + 2r$.



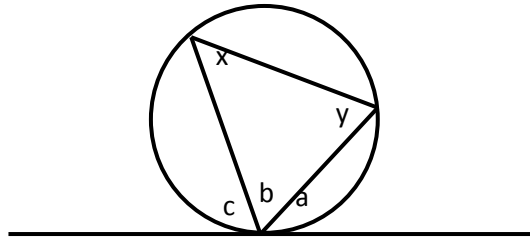
In Triangle OPA, OA is perpendicular to PA, thus
 $(R + r)^2 = x^2 + (R - r)^2$
 $P = (R + r) + x + (R - r)$
 $A = \frac{1}{2} OA \cdot PA$
 $= \frac{1}{2} x \cdot (R - r)$

If we define radius of circle O is R and radius of circle P is r , then the maximum height is $2R$ for the externally tangent circles. Distance between centres is $R + r$.

Inscribed Triangle and a Tangent



Angle made by a tangent and a chord is equal to the opposite interior angle.



$$x + y + b = 180^\circ$$

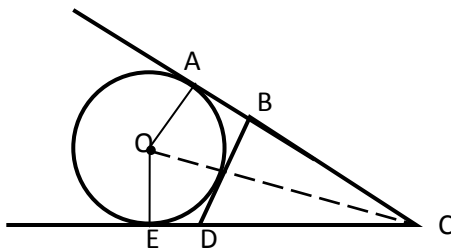
$$a + b + c = 180^\circ$$

$$a = x$$

$$c = y$$

Escribed Circle, Ecirle or Excircle

A circle tangent to one side of a triangle and to the extensions of the other two sides of a triangle, the centre of which is **excentre**, and radius is **exradius**.



From $\triangle BCD$ we have drawn circle O and
O is excentre

OA and OE are exradii

BD, AC and EC are tangents

$$AC^2 + OA^2 = OC^2$$

$$EC^2 + OE^2 = OC^2$$

Inscribed Circle

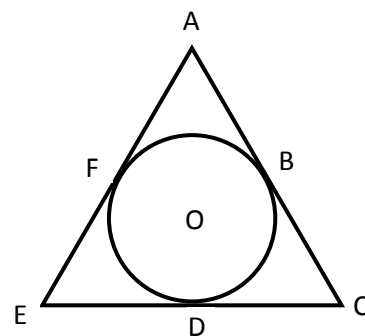
Equilateral Triangle ACE **circumscribes** circle O.

Circle O is **inscribed** in triangle ACE. The points of Tangency are B, D and F. Chords BD, DF and BF form another equilateral triangle.

Bisector of triangle ruled from any vertex bisects the circle and is the median of the triangle.

Triangles ABF, BCD, DEF and BDF are congruent.

And line segments OA, OC and OE are congruent. $\square OBAF$, $\square OBCD$ and $\square ODEF$ are congruent kites.

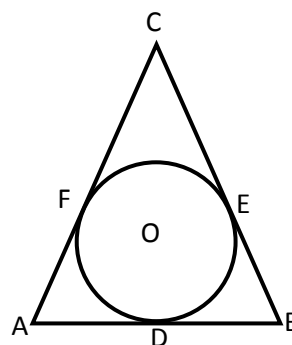


Note that **medians** of triangles are line segments ruled from a vertex to the mid-point of the opposite side to the vertex.

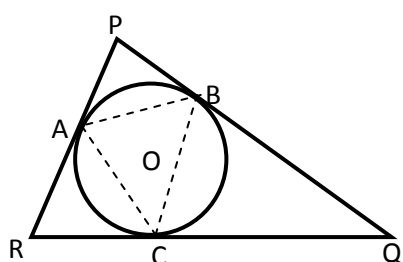


Isosceles Triangle ABC circumscribes circle O.
Circle O is inscribed in triangle ABC. The points of tangency are D, E and F.

Bisector of triangle ruled from vertex C bisects the circle and is the median of the triangle at D. Points E and F do not form medians with respective vertices.



Triangles ADF, BDE and DEF are congruent since $\square ABEF$ is an isosceles trapezium.



The Scalene triangle PQR circumscribes circle O. A, B and C are points of tangency. Medians of the triangle may not bisect the circle nor contain point of tangency.

Circle and two Tangents

The circle below is not an ecircle because there is no tangent triangle drawn externally.

In the diagram, OA and OB are radii, AB is a chord. PA and PB are tangents and PO bisects $\square PAOB$ to form two right triangles $\triangle PAO$ and $\triangle PBO$ where $\triangle PAO \cong \triangle PBO$. And similarly we can say $\triangle PAD \cong \triangle PBD$ and $\triangle AOD \cong \triangle BOD$. PO also bisects minor arc ACB, minor segment ACB, chord AB and sector OACB. $\triangle APB$ and $\triangle AOB$ are isosceles

Since $\triangle PAO$ and $\triangle PBO$ are RAT

$$PO^2 = OA^2 + PA^2 \text{ and}$$

$$PO^2 = OB^2 + PB^2$$

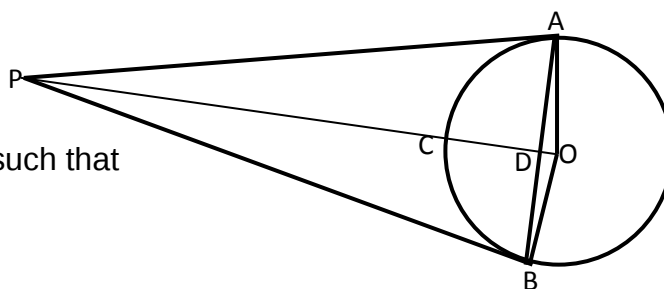
Inscribed angle not indicated is such that

$$\angle ACB = \frac{1}{2} \angle \text{arc AB}$$

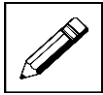
Other angles

$$\angle PAD = \angle PBD, \angle AOD = \angle BOD, \angle APC = \angle BPC$$

$$\angle CDB = \angle CDA = \angle ODB = \angle ODA = \angle OAP = \angle OBP = 90^\circ$$



We can deduce similar properties in escribed circle only when the externally drawn triangle is an isosceles triangle and point of tangency happens to be the midpoint of the tangent side.

**LEARNING ACTIVITY 11.4.3.4 to 11.4.3.5**

20 minutes

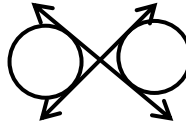
A. Multiple Choice - Write the letter of your choice before each number.

_____ 1. Which of the following are considered internally tangent circles?

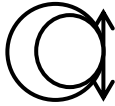
a.



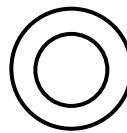
c.



b.



d.



_____ 2. P is a point in the exterior of a circle. How many tangents to the circle may contain point P ?

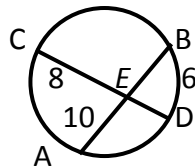
a. 1

b. 2

c. 3

d. 4

_____ 3. Chords $\overline{AB}$ and $\overline{CD}$ intersect at centre E . If $\overline{AE} = 10$, $\overline{EB} = 6$, $\overline{CE} = 8$, and $\overline{ED} = x$, what is $\overline{CD}$? Use the figure below.



a. 15.5

b. 8

c. 7.5

d. 4

_____ 4. Two or more circles with the same centre

a. congruent circles

c. equal circles

b. concentric circles

d. centre circles

_____ 5. An arc that is one half of the circle is

a. sphere

c. semicircle

b. hemisphere

d. intercepted arc

_____ 6. The point of contact when a line passes through a circle at exactly one point is

a. point of concurrency

c. point of no return

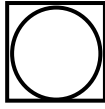
b. point of tangency

d. point of origin

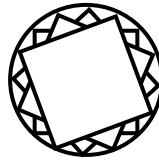


_____ 7. Which of the following is a cyclic quadrilateral?

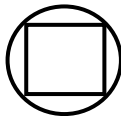
a.



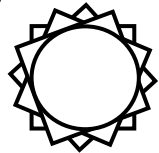
c.



b.



d.



_____ 8. In a cyclic quadrilateral, the quadrilateral is _____ in a circle.

a. circumscribed

c. inscribed

b. combined

d. traced

_____ 9. "A quadrilateral is said to be a cyclic quadrilateral if there is a quadrilateral passing through all of its four vertices." This statement is

a. True

c. Either True or False

b. False

d. Neither True nor False

_____ 10. If a line is tangent to a circle, then it is _____ to the radius drawn to the point of tangency.

a. equal

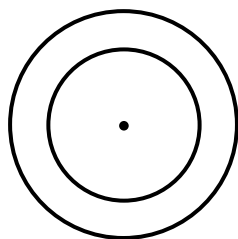
c. tangent

b. perpendicular

d. parallel

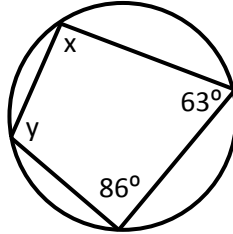
B. Solve the following problems.

1. Calculate area between the concentric circles, the annulus as given below using the formula $A = \pi R^2 - \pi r^2$ when $R = 4$ cm and $r = 3$ cm.

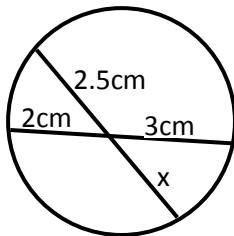




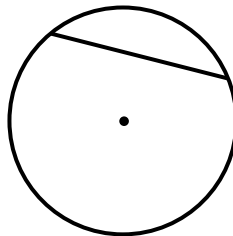
2. Find the measures of angles marked x and y in the cyclic quadrilateral.



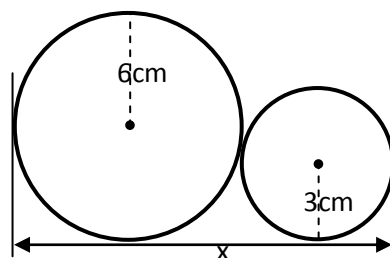
3. Calculate measure of segment x .



4. Calculate length of chord y correct to two decimal places given that, the diameter is of the circle is 4 cm, and the chord is 1.5 cm away from the centre.



5. Study the diagram below and answer questions that follow.



a. What is the distance between the centres?

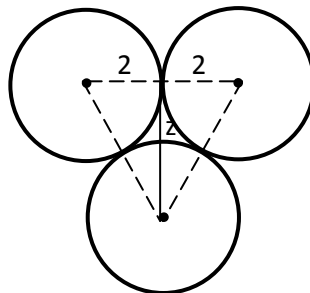


b. What is the difference between the radii?

c. Calculate horizontal distance between the centres or centre points.

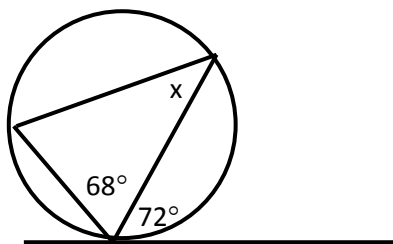
d. Find x .

6. Use Pythagoras to solve for z (height of triangle) in the congruent externally tangent circles.



7. Construct congruent circles which are externally tangent, with radii of 3cm. Rule a rectangle whose sides are tangents to either or both circles. Calculate area of rectangle not covered by congruent circles.

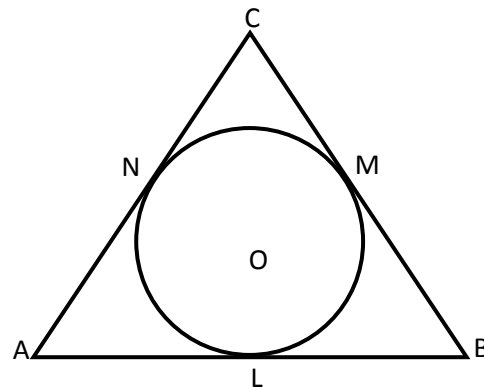
8. Find value of angle x .





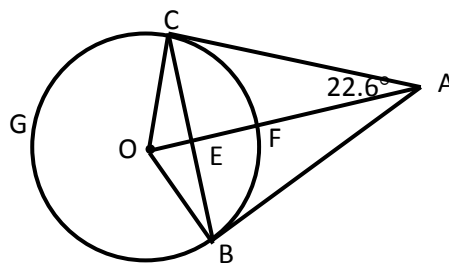
9. Compute the measurements given that ABC is equilateral triangle, and circle O is inscribed in triangle ABC. AB is 12 cm.

- Chord LM = _____
- Line Segment AN = _____
- Altitude CL = _____
- Angle LMN = _____
- Angle NOM = _____



10. Study the diagram and answer the following questions. Given that $OG = 5$ cm, $AB = 12$ cm, $\angle CAO = 22.6^\circ$ and $\square OCAB$ is a kite.

- $\angle ACO =$ _____
- $\angle AEC =$ _____
- $\angle COA =$ _____
- $\angle OBE =$ _____
- $\angle EBA =$ _____
- $\angle BOC =$ _____
- Reflex $\angle BOC =$ _____
- $\angle BGC =$ _____
- $OA =$ _____
- $OF =$ _____
- $OE =$ _____
- $BC =$ _____
- $CE =$ _____
- Arc length BF = _____

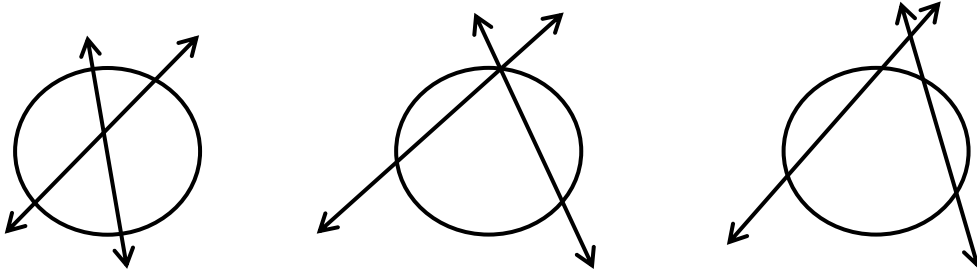




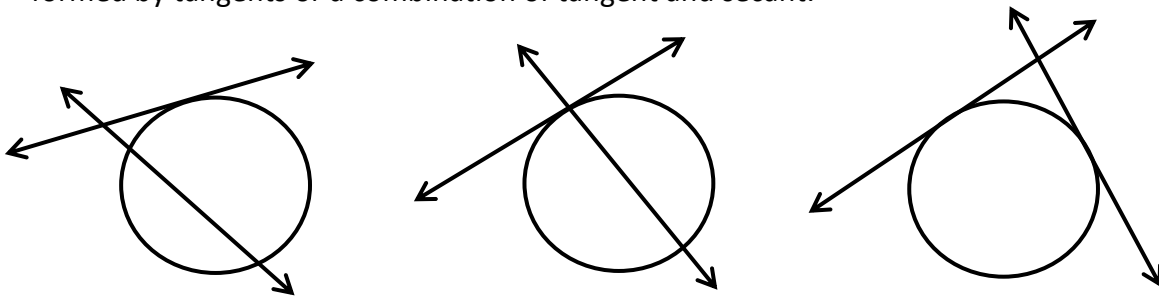
11.4.3.6 The Angles formed by Chords, Secants and Tangents

Note that not only are central angles formed by points on the circle and not only central angles will intercept arcs. You may have also noticed that central angles and/or inscribed angles may be formed using chords, secants, and tangents.

Observe how angles are formed in the circles as shown below.



You may notice that the angles are formed by secants whose vertices are inside, outside and on the circle; thus, they are called **secant angles**. On the other hand, the angles below are formed by tangents or a combination of tangent and secant.

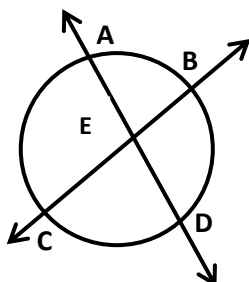


As shown in the figures above, angles formed by a secant and a tangent are called secant-tangent angles and the angle formed by two tangents is called tangent-tangent angle. The following theorems will tell us how to find the measure of such angles.

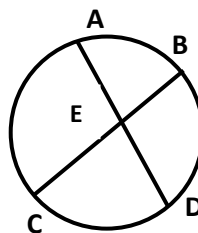
Secant-Secant Theorem

If two secants intersect in the interior of a circle, then the measure of an angle formed is equal to one-half the sum of the measures of the intercepted arcs. This theorem is also true for chords.

Example If $m\angle AEC$ is 110° and $m\widehat{AC}$ is 100° , find $m\widehat{BED}$ and $m\widehat{BD}$.



or





Solution

Since $\angle AEC$ and $\angle BED$ are vertical angles, they have equal measures, that is,
 $m\angle AEC = m\angle BED = 110$.

To find the measure of arc BD , we use the Secant-Secant Theorem, that is,
 $m\angle AEC = \frac{1}{2} (m\widehat{AC} + m\widehat{BD})$.

$$110 = \frac{1}{2} (100 + m\widehat{BD})$$

$$110 = \frac{1}{2} (100) + \frac{1}{2} \widehat{BD}$$

$$110 = 50 + \frac{1}{2} \widehat{BD}$$

$$110 - 50 = \frac{1}{2} \widehat{BD}$$

$$60 = \frac{1}{2} \widehat{BD}$$

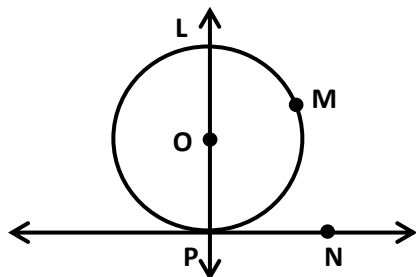
$$120 = \widehat{BD}$$

Therefore $m\angle BED = 110$ and $m\widehat{BD} = 120$.

Secant-Tangent Theorem (Case 1)

When a tangent and a secant (or chord) intersect in a point on a circle, the measure of the angle formed is one-half the degree measure of the intercepted arc

Example



In the given figure, secant LP that passes through the centre of circle O intersects tangent PN at point P as the point of tangency. Prove that $\angle LPN = \frac{1}{2} \widehat{LMP}$.

Solution:

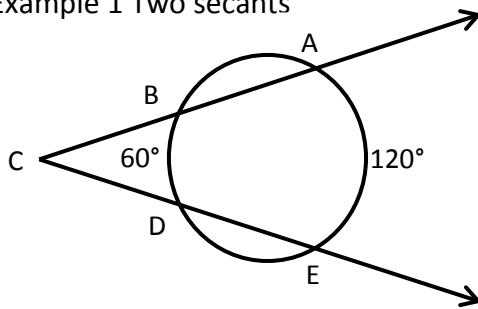
$\overrightarrow{LP} \perp \overrightarrow{PN}$ since according to Tangent Line Theorem, a line tangent to a circle is perpendicular to the radius (or diameter) of the circle. Thus $\angle LPN = 90^\circ$. $\widehat{LMP}$ is a semicircle so $\widehat{LMP} = 180^\circ$. By transitive property of equality, we may conclude that $\frac{1}{2} \widehat{LMP} = 90^\circ$ and $\angle LPN = \frac{1}{2} \widehat{LMP}$.

Secant-Tangent Theorem (Case 2)

When two secants, a tangent and a secant or two tangents intersect in the exterior of a circle, the measure of the angle formed is one-half the difference of the measures of the intercepted arc.

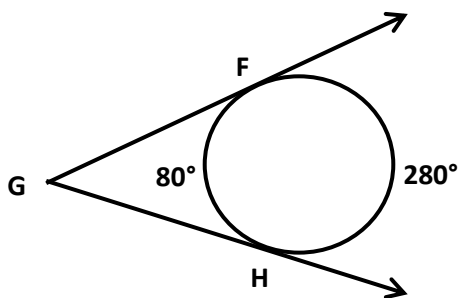


Example 1 Two secants



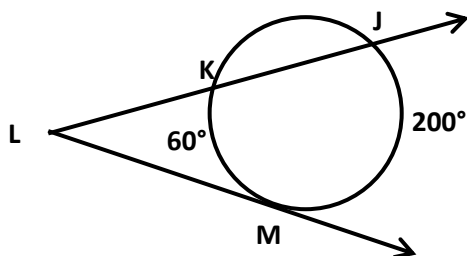
$$\begin{aligned}\angle C &= \frac{1}{2} (\widehat{AE} - \widehat{BD}) \\ \angle C &= \frac{1}{2} (120^\circ - 60^\circ) \\ \angle C &= \frac{1}{2} (60^\circ) \\ \angle C &= 30^\circ\end{aligned}$$

Example 2 Two tangents

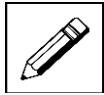


$$\begin{aligned}\angle G &= \frac{1}{2} (\widehat{FIH} - \widehat{FH}) \\ \angle G &= \frac{1}{2} (280^\circ - 80^\circ) \\ \angle G &= \frac{1}{2} (200^\circ) \\ \angle G &= 100^\circ\end{aligned}$$

Example 3 One secant, one tangent



$$\begin{aligned}\angle L &= \frac{1}{2} (\widehat{JM} - \widehat{KM}) \\ \angle L &= \frac{1}{2} (200^\circ - 60^\circ) \\ \angle L &= \frac{1}{2} (140^\circ) \\ \angle L &= 70^\circ\end{aligned}$$



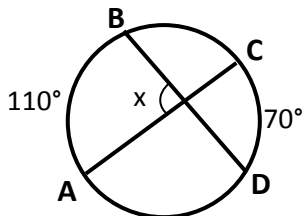
LEARNING ACTIVITY 11.4.3.6



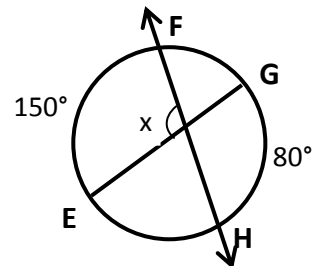
20 minutes

A. Find the measure of $\angle x$ in each of the following.

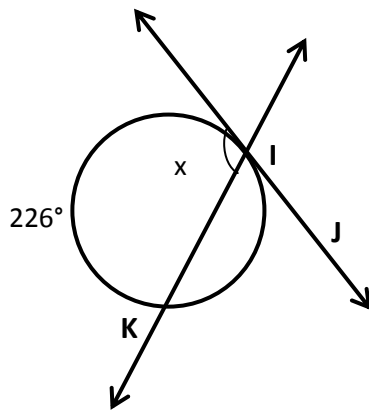
a.



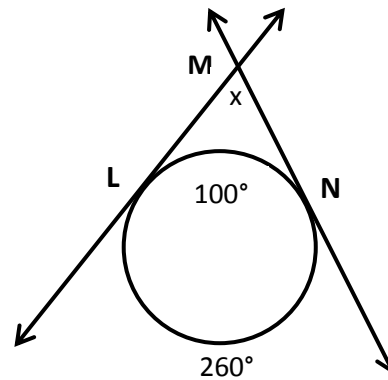
b.



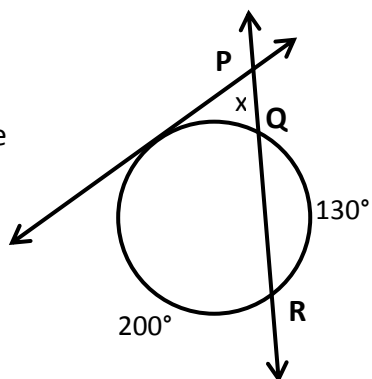
c.



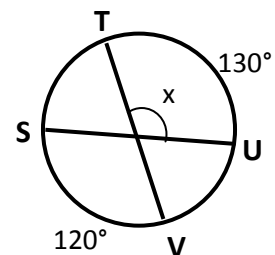
d.



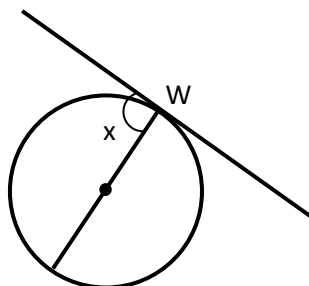
e.



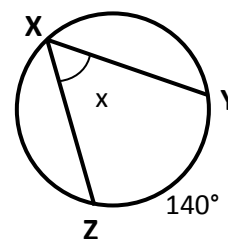
f.



g.



h.



**SUMMATIVE TASK 11.4.3**

20 minutes

A. Identification

Write the correct answer before each number.

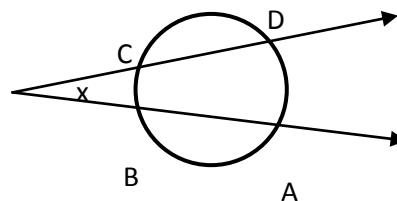
- _____ 1. What is the longest chord in a circle?
- _____ 2. What figure is formed by the intersection of a secant and a circle?
- _____ 3. It refers to the point of contact when a line passes through a circle at exactly one point?
- _____ 4. If a circle has a radius of one (1) and its centre is at the origin of a Rectangular Coordinate Plane, what is the equation of the circle?
- _____ 5. What is the measure of the longest chord in a unit circle
- _____ 6. A secant-tangent angle has its vertex _____ the circle.
- _____ 7. The measure of two angles that are equal and supplementary.
- _____ 8. A geometric figure which refers to the line segment joining the centre and one point on the circle?
- _____ 9. A curve line which is a part of a circle included between any of its points.
- _____ 10. an arc which is one half of the circle

B. Show your complete solution in solving the following:

1. Find angle x when:

a. $\widehat{AD} = 108^\circ$ and $\widehat{BC} = 42^\circ$

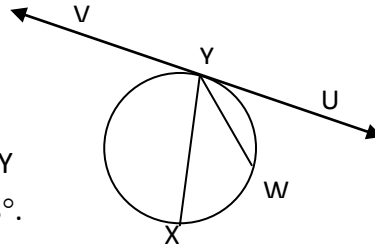
b. $\widehat{AD} = 117^\circ$ and $\widehat{BC} = 35^\circ$



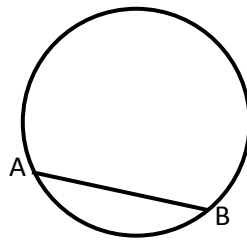


2. In the given figure at the right,

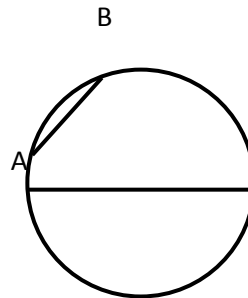
In the diagram VU is tangent at Y. WY and XY are chords such that $\angle WYX = 72^\circ$ and $\angle XWY = 88^\circ$. Find $\angle UYW$, $\angle UYX$ and $\angle WYX$.



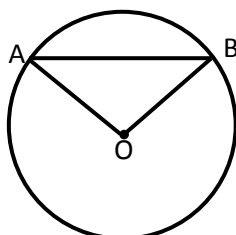
3. Find the radius of the circle if the chord AB is 2 cm and is 1 cm from the centre.



4. What is the length of the chord AB when the radius is 5 cm and 4 cm away from the centre?

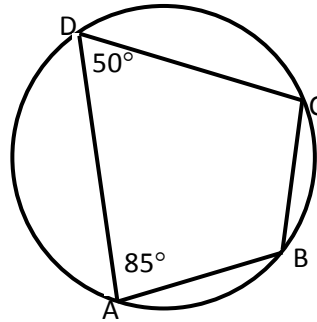


5. Calculate area of triangle OAB if OA is 5 cm and AB is 8 cm long.

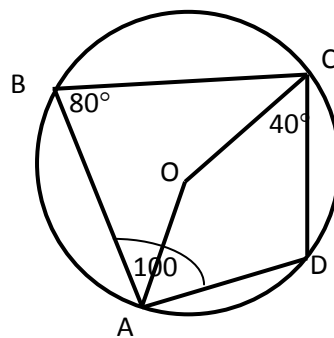




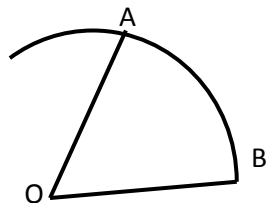
6. Calculate angle ABC.



7. Calculate angle OAD.



8. Calculate area of triangle OAB given that AB is 60° and OA is 10 cm.





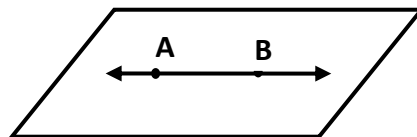
11.4.4: SOLID GEOMETRY

Most of the figures that we see everyday are solid figures like cubes, prisms, cones, etc. These solid figures cannot be formed without plane figures. In Euclidian Geometry, **plane geometry** is about two-dimensional figures while **solid geometry** involves three-dimensional figures. Since measurements of three-dimensional objects is covered in this topic, we will revise what we learnt on plane geometry.

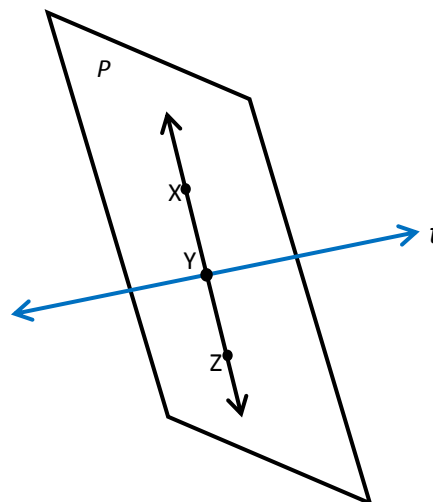
11.4.4.1 Lines and Planes

A two-dimensional plane figure can be formed using undefined terms. A **line** as we learnt earlier is a set of continuous points that extends indefinitely in either direction. You have learnt that a line has one dimension only, that it has length but no width nor thickness.

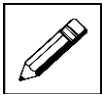
In plane geometry, we always refer to straight line only but in solid geometry, a line can be straight or curved or a combination. Another undefined term is about plane is a flat surface. Compared to line, a plane has two dimensions for it has length and width but has no height nor thickness. A straight line connecting any two of its points lies completely in a plane. These two points illustrated in the figure below are collinear and coplanar at the same time.



Notice the collinear and coplanar points X, Y, and Z in plane P . A theorem on perpendicular line states that: In a plane, at a given point in a line, there is exactly one line perpendicular to the line.



From the given figure, $\overleftrightarrow{XZ} \perp$ to line t at point Y. We may also draw a line perpendicular to point X or point Z. This indicates that a perpendicular line may pass through any given point on the plane.

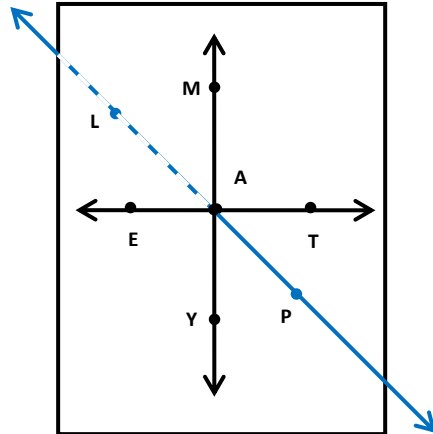


LEARNING ACTIVITY 11.4.4.1



10 minutes

In the plane figure, $\overleftrightarrow{MY} \perp \overleftrightarrow{ET}$ at point A. If $\overleftrightarrow{LP}$ is drawn perpendicular at point A, classify the following statements as TRUE or FALSE.



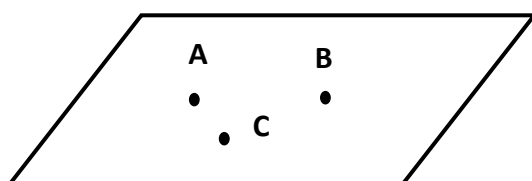
1. $\overleftrightarrow{LP} \perp \overleftrightarrow{ET}$
2. $\overleftrightarrow{LP} \perp \overleftrightarrow{MY}$
3. $\angle PAT$ is an acute angle
4. $\angle PAY$ is a right angle
5. $\angle MAL$ and $\angle PAT$ are vertical angles
6. Points M and P are coplanar points
7. $\angle MAT$ and $\angle PAT$ are supplementary angles
8. $\angle MAP$ is an obtuse angle
9. $\angle TAP$ is a right angle
10. $\angle MAT \cong \angle PAT$



11.4.4.2 The Angle Between a Line and a Plane

An angle is formed by two rays having the same endpoint. Although plane figures are formed using line segments, these figures can also form angles. Postulates, principles and theorems involving planes are discussed in this lesson.

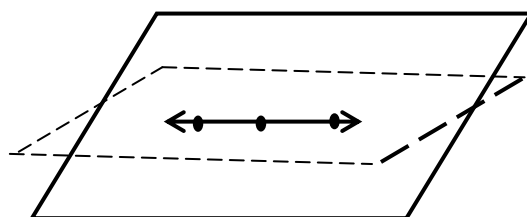
Three points on a plane postulate: A plane contains at least three noncollinear points.



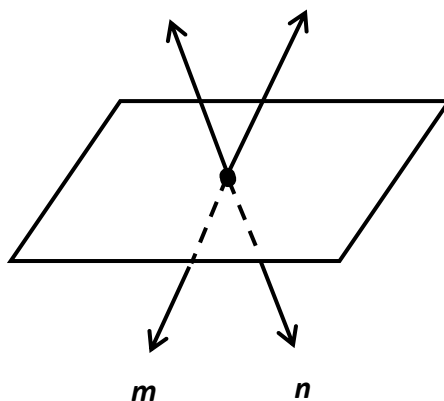
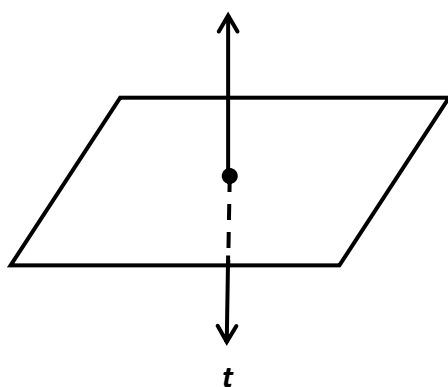
On the other hand, through any three noncollinear points, there is only one plane. If the three points are collinear, there is at least one plane. This means that these three collinear points can be contained in more than one plane.

Refer to the figure below to understand why we use the statement “*at least one plane*”.

Intersection of planes postulate: If two planes intersect, their intersection is a line.



Note that the intersection between a line and a plane is a point. Hence, angles can be formed through the point of intersection as shown in the figures below. The angle formed depends on how the line will pass through a plane.

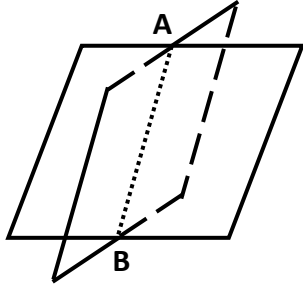


In the first figure, notice that line t is perpendicular to a plane and therefore forms a right angle. In the other figure, lines m and n are not perpendicular to a plane which therefore form either acute or obtuse angles.



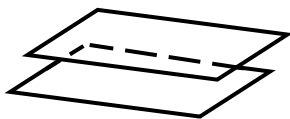
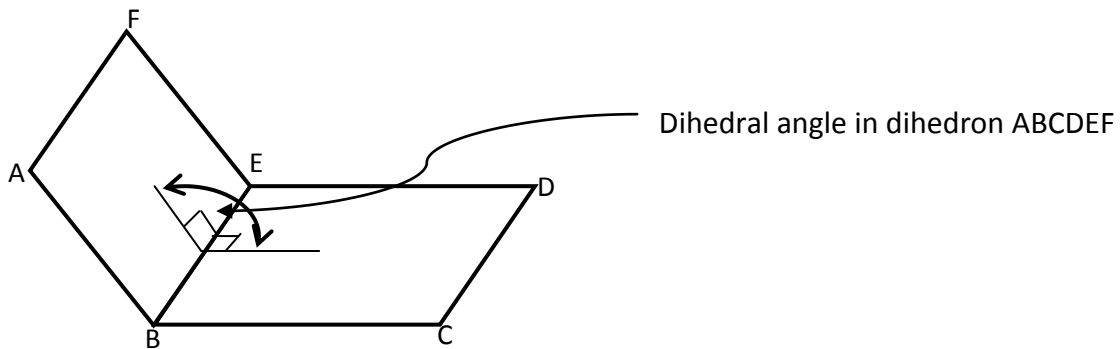
11.4.4.3 The Angle Between Two Planes

Angles can also be formed between two planes. If the intersection between a line and a plane is a point, the intersection between two planes is a line.



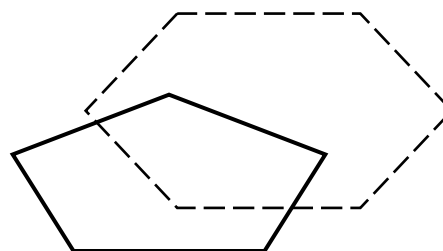
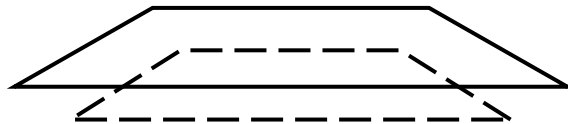
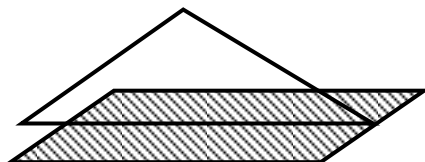
The figure shows segment AB is the intersection of the two planes as described in the postulate above.

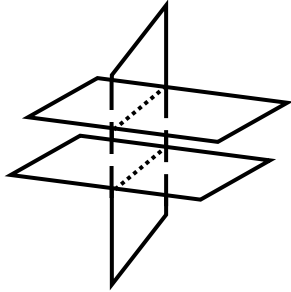
The intersection of two planes is called **dihedron**. The angle so formed by the two planes is a **dihedral angle**. In the intersecting planes above, there are four dihedral angles. If they are intersected at the edge, there will be two dihedral angles.



Parallel planes are planes that do not intersect.

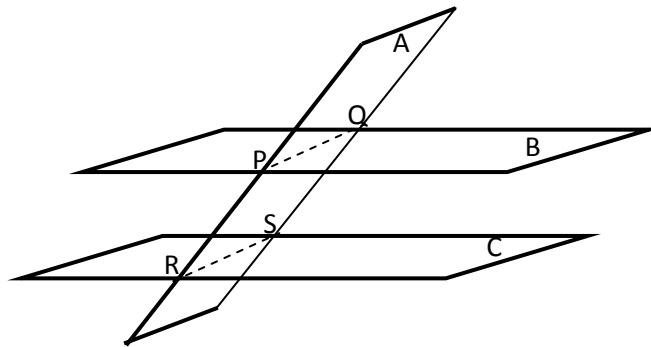
The two or more of Parallel planes do not necessarily have to be in the same shape, nor are congruent. Parallel planes can be observed in solid figures, especially the prism.





If two parallel planes are intersected by a third plane, then the lines of intersection are parallel. The angles formed may be compared to special angle pairs.

Example Plane A intersects planes B and C at PQ and RS respectively. Which angle is co-interior to $\angle CSQ$?



Solution

Since $\angle CSQ + \angle BQS = 180^\circ$

Therefore $\angle BQS$

Note that the planes were defined with a single letter, thus making it difficult to define angles.

**LEARNING ACTIVITY 11.4.4.2 to 11.4.4.3**

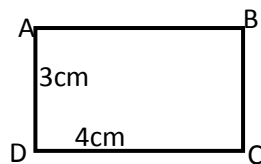
10 minutes

A. TRUE or FALSE. Write True if the statement is correct otherwise write False.

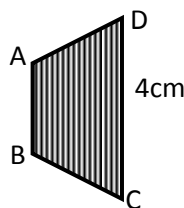
- _____ 1. The intersection between two planes is a line.
- _____ 2. A plane contains at most three non-collinear points.
- _____ 3. Three collinear points determine a plane.
- _____ 4. The intersection between a line and a plane is a point.
- _____ 5. If two parallel planes are intersected by a third plane, then the lines of intersection are parallel.
- _____ 6. Through any three noncollinear points, there is only one plane.
- _____ 7. Three collinear points can be contained in more than one plane.
- _____ 8. In a plane, at a given point in a line, there is exactly one line perpendicular to the line.
- _____ 9. Two planes may intersect at one point.
- _____ 10. A line may intersect two or more planes.

B. Problems

1. If the rectangular plane **ABCD** has an axis at **AD**, and is rotated about **AD** will produce a cylinder after a complete revolution. Calculate the volume using $v = \pi r^2 h$.

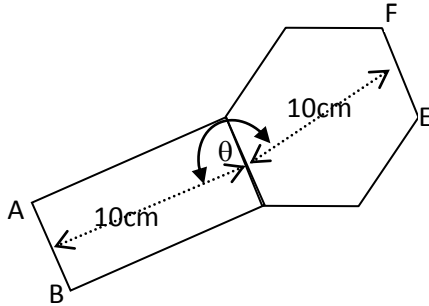


2. If a trapezoidal plane **ABCD** has an axis at **AB**, and is rotated about **AB** will produce two cylinders after a complete revolution. Calculate the total (including space) volume using $v = \pi r^2 h$. Perpendicular distance of AB and DC is 2 cm.





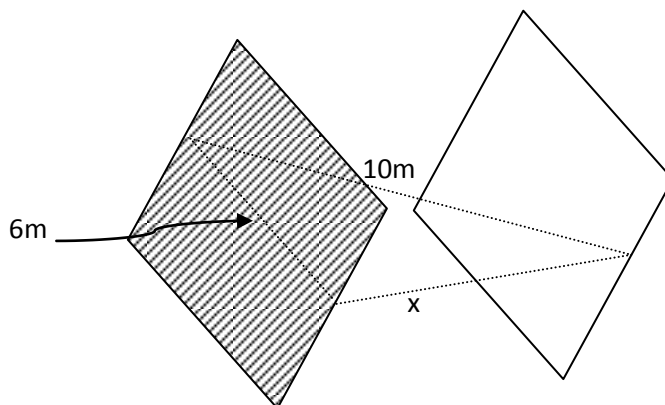
3. Calculate dihedral angle marked θ below, given that horizontal distance between AB and EF is 6 cm. Use Cosine Rule $a^2 = b^2 + c^2 - 2bc \cos A$.



4. Draw a plane parallel to plane given.



5. Use Pythagoras Theorem to calculate the horizontal distance x between the two parallel planes. $c^2 = a^2 + b^2$





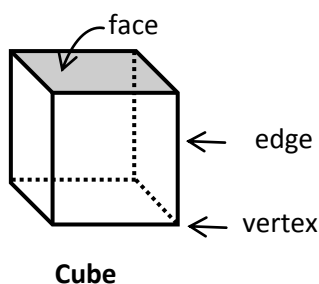
11.4.4.4 Calculations from 3D Figures

Most products that we see in supermarkets are packed in different forms like boxes, cans, bottles, tetra packs, and the like. These packages show common characteristics of three-dimensional or solid figures. When we think of area and perimeter, we usually think of 2D shapes or plane figures.

On the other hand, when we think of volume and surface area, we usually relate them to 3D shapes or solid objects. A three-dimensional object is a solid figure bounded by intersecting planes called **polyhedron**. It is formed by a number of polygonal regions called the **faces** of the polyhedron. The edges of the faces formed at the intersection of planes are called the **edges** and the intersection of these three or more edges is a point called **vertex**. Common examples of polyhedra are cubes, prisms, and pyramids.

A **prism** is a polyhedron with two opposite faces that are congruent and parallel. The other faces are parallelogram. The following are examples of a prism:

CUBOID



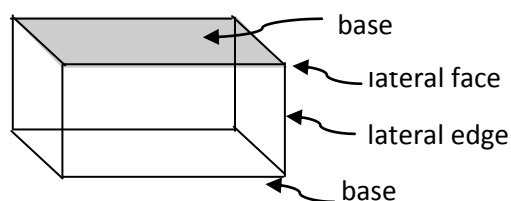
$$SA = 6s^2$$

$$V = s^3$$

If base area is known, we can use

$$V = Ah \text{ where } A \text{ is base area}$$

RECTANGULAR PRISM



$$SA = 2(lw + lh + wh)$$

$$V = lwh$$

If the base area is known, we can use

$$V = Ah \text{ where } A \text{ is the base area.}$$

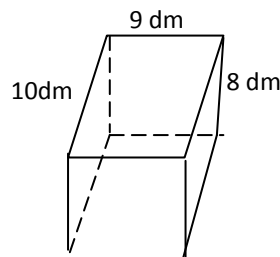
$$V = Ph + 2A \text{ where } P \text{ is the perimeter of base area, } A \text{ is base area and } h \text{ is the perpendicular height to the base face}$$

Example 1 Find the lateral surface area, total surface area and volume of a rectangular prism whose base dimensions are 9 dm and 10 dm and whose altitude is 8 dm.



Solution

First, we need to draw and label the figure of the rectangular prism.



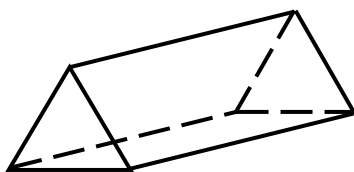
To find the lateral surface area, we use $L = aP$. The perimeter of the base that is rectangular is computed as $P = 2L + 2W$. Since the length is 10 dm and the width is 9 dm, the perimeter of the base is $P = 2(10) + 2(9)$; therefore, $P = 38$ dm. Using $L = aP$, we substitute the values of the altitude $a = 8$ dm and the obtained perimeter = 38 dm. The lateral surface area is $L = 8 \text{ dm}(38 \text{ dm})$; $L = 304 \text{ dm}^2$.

Next, to find the total surface area, we use $T = L + 2B$. Though we have $L = 304 \text{ dm}^2$, we have to compute first the area of the base, that is, $B = LW$ (formula for the area of rectangle since the base is rectangular) to be able to compute the total surface area.

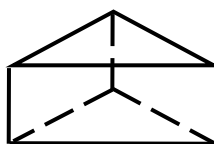
$B = (10 \text{ dm})(9 \text{ dm})$; $B = 90 \text{ dm}^2$. Now, we substitute the values of L and B in the formula $T = L + 2B$. $T = 304 \text{ dm}^2 + 90 \text{ dm}^2$, $T = 394 \text{ dm}^2$.

Lastly, to find the volume, we use $V = aB$. Substitute $a = 8 \text{ dm}$ and $B = 90 \text{ dm}^2$, we get $V = 8 \text{ dm}(90 \text{ dm}^2)$; $V = 720 \text{ dm}^3$.

Therefore, $L = 304 \text{ dm}^2$, $T = 394 \text{ dm}^2$ and $V = 720 \text{ dm}^3$.



Triangular Prism



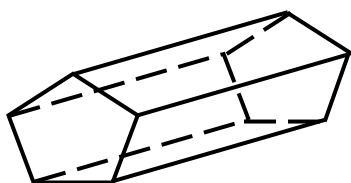
Triangular Prism

$SA = Ph$, P is perimeter of base face

$V = \frac{1}{2} lwh$

If base area is known (triangular $V = Ah$ and rectangular use $V = \frac{1}{2} Ah$) where A is base area.

Pentagonal Prism



The examples of prisms above are named according to the shape of its base. The lateral faces are rectangular.



Below are the formula in finding the volume, lateral surface area and total surface area of prisms.

Lateral Area

$$L = aP$$

Total Surface Area

$$T = L + 2B \quad \text{SA}$$

Volume

$$V = aB / V = Ah$$

Where L – lateral surface area

a – altitude or height

P – perimeter of the base

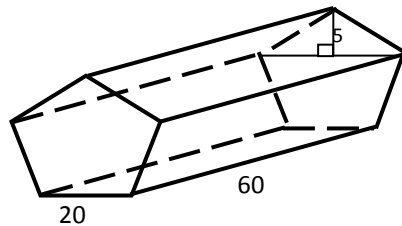
T – total surface area

B – area of the base (area computed according the polygonal shape of the base)

V – volume

Let us do some calculations of prisms.

Example 2 Given the dimensions of a regular pentagonal prism in meters, find the lateral area, total surface area and volume.



Solution

a) Lateral Area

Since the measure of one side of a regular pentagonal base of the prism is 20 m and the altitude is 60m, we compute the lateral area.

$$L = aP$$

$$L = 60m (5S)$$

$$L = 60m (5 \cdot 20m)$$

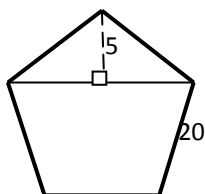
$$L = 60m(100m)$$

$$L = 6000m^2$$

b) Total Surface Area

First, we compute the area of the base (B) before finding the total surface area. You may notice that the regular pentagonal base may be divided into two regions:

A trapezium and a triangle as shown below.



$$T = L + 2B, \text{ or}$$

$$T = L + 2(A_{\text{triangle}} + A_{\text{trapezium}})$$

$$\text{since } B = A_{\text{triangle}} + A_{\text{trapezium}}$$



$$A_{\text{triangle}} = \frac{1}{2}bh$$

To find the base, we use **the** Pythagorean Theorem $c^2 = a^2 + b^2$

We let $c = 20$ and $a = 5$

$$b^2 = c^2 - a^2$$

$$b = \sqrt{(20)(20) - (5)(5)}$$

$$b = \sqrt{400 - 25}$$

$$b = \sqrt{375}$$

$$b = 19.36 \text{ m}$$

But we need to multiply b to 2 because only $\frac{1}{2}$ of the base was computed using Pythagorean Theorem in order for us to find the area of the triangle.

b (base of the triangle) = $2(19.3649\text{m})$

$b = 38.73 \text{ m}$, and since $h = 5\text{m}$

$$A_{\text{triangle}} = \frac{1}{2}bh$$

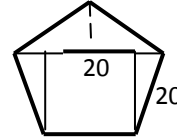
$$A_{\text{triangle}} = \frac{1}{2}(38.73\text{m})(5\text{m})$$

$$A_{\text{triangle}} = 96.825\text{m}^2 \text{ or}$$

$$A_{\text{triangle}} = 96.825 \text{ m}^2$$

$$A_{\text{trapezoid}} = \frac{1}{2}(b_1 + b_2)h$$

The base of the triangle $b = 96.82\text{m}$ is our b_1 and $b_2 = 20\text{m}$. To find the height of the trapezoid, we subtract 20 from $b = 38.73\text{m}$



$$a = (38.73 - 20)/2$$

$$h^2 = c^2 - a^2$$

$$h = \sqrt{(20)(20) - (9.365)(9.365)}$$

$$h = \sqrt{400 - 87.70}$$

$$h = \sqrt{312.30}$$

$$h = 17.67 \text{ m}$$

$$A_{\text{trapezium}} = \frac{1}{2}(b_1 + b_2)h$$

$$A_{\text{trapezium}} = \frac{1}{2}(96.82 + 20)17.67$$

$$A_{\text{trapezium}} = 1032.22 \text{ m}^2$$

$$T = L + 2(A_{\text{triangle}} + A_{\text{trapezium}})$$

$$T = 6000\text{m}^2 + 1032.22 \text{ m}^2$$

$$T = 7032.22 \text{ m}^2$$

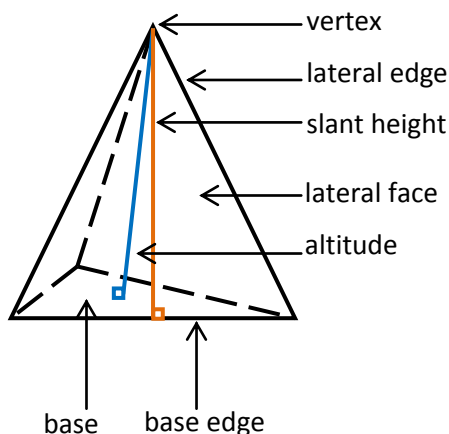
c. Volume

$$V = aB$$

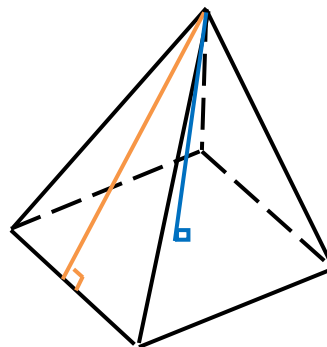
$$V = 60\text{m}(1032.22 \text{ m}^2)$$

$$V = 61933.01 \text{ m}^3$$

Let us now study other polyhedra in the form of a pyramid. A **pyramid** is a polyhedron whose lateral faces have one common vertex. Just like the prism which can be named according to the shape of the parallel polygonal bases, a pyramid can be named according to the shape of the base. Below are examples of triangular pyramid and a square pyramid with their corresponding parts.



Triangular Pyramid



Square Pyramid



The following are the parts of the pyramid:

The **base edge** is the intersection of the base and a lateral face.

The **base** is a face that does **not contain the common vertex**.

The **altitude** is a segment drawn perpendicular from the common vertex to the base.

The **slant height** is the altitude of a lateral face of the pyramid.

The pyramid is computed by the following formula

Volume, $V = \frac{1}{3} aB$

where L – lateral surface area

l – slant height

Lateral Area, $L = \frac{1}{2} l P$

P – perimeter of the base

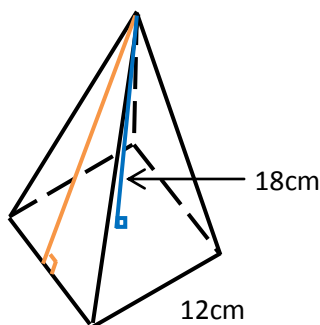
Total Surface Area, $T = L + B$,

B – area of the base

Example1 A pyramid has a square base of 12 cm on a side and an altitude of 18 cm high.

Find the lateral and total surface areas and volume.

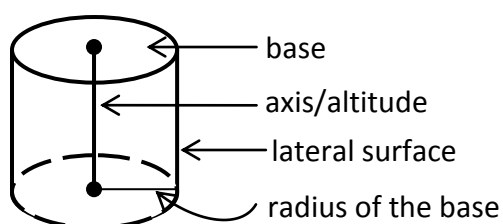
Solution:



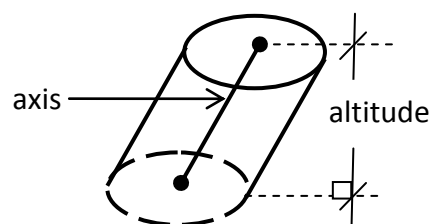
Lateral Area	Total Surface Area	Volume
$L = \frac{1}{2} l P$	$T = L + B$	$V = \frac{1}{3} aB$
By Pythagorean Theorem	$B = s^2$ (Area of base)	
$l^2 = (18\text{cm})^2 + (6\text{cm})^2$	$B = (12\text{cm})^2$	$V = \frac{1}{3} (18\text{cm})(144\text{cm}^2)$
$l = \sqrt{360} \text{ cm}^2$	$B = 144 \text{ cm}^2$	$V = 864 \text{ cm}^3$
$l = 18.97 \text{ cm}^2$	$T = 455.37\text{cm}^2 + 144 \text{ cm}^2$	
$P = 4 (12\text{cm})$	$T = 599.37 \text{ cm}^2$	
$P = 48 \text{ cm}$		
$L = \frac{1}{2} (18.97\text{cm})(48\text{cm})$		
$L = 455.37 \text{ cm}^2$		

The figures we have just discussed are polyhedrons which are solid figures. Other kinds of solid figures are cylinder, cone and sphere. Unlike prisms and pyramids that are formed by closed polygonal figures; cylinder, cone and sphere have different curved surfaces on it.

A **cylinder** is a space figure whose bases are congruent and are parallel circles. A cylinder is **right** if the axis is also its altitude ; otherwise, it is **oblique**. The **altitude** of a cylinder is a segment perpendicular to the bases. The **axis** is a segment whose endpoints are the centers of the bases.



Right Cylinder



Oblique Cylinder



Volumes and Surface Areas

$$V = \pi r^2 h$$

$$SA = 2\pi r^2 + 2\pi rh$$

Example 1 Calculate volume of an oblique cylinder with radius of 3 cm and height of 5 cm.

Solution

$$V = \pi r^2 h$$

$$= 3.14 \times 3^2 \times 5$$

$$= 3.14 \times 45$$

$$= 141.1 \text{ cm}^3$$

Example 2 Calculate surface area of a right cylinder with radius of 7 cm and height of 5 cm.

Use $\pi = 22/7$.

Solution

$$SA = 2\pi r^2 + 2\pi rh$$

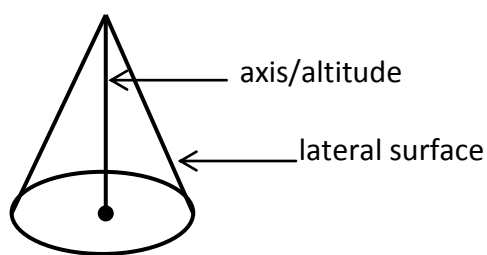
$$= 2 \times 22/7 \times 7^2 + 2 \times 22/7 \times 7 \times 5$$

$$= 2 \times 22 \times 7 + 2 \times 22 \times 5$$

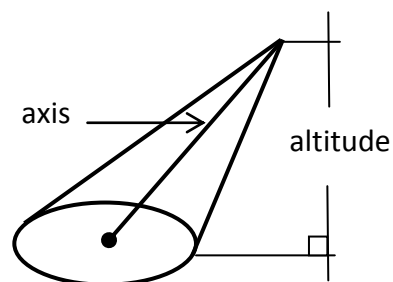
$$= 308 + 220$$

$$= 528 \text{ cm}^2$$

Just like the right and oblique cylinders, a cone can be right or oblique too. If a cylinder has two parallel and congruent circular bases, a cone has only one circular base.



Right Cone



Oblique Cone

Volumes and Surface Areas

$$V = \frac{1}{3} \pi r^2 h$$

$$SA = \pi r^2 + \pi rh$$



Example 1 Calculate volume of a cone with radius of 3 cm and a height of 7 cm.

Solution

$$\begin{aligned} V &= \frac{1}{3} \times \frac{22}{7} \times 3 \times 3 \times 7 \\ &= 22 \times 3 \\ &= 66 \text{ cm}^3 \end{aligned}$$

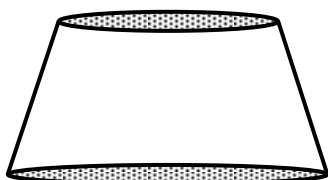
Example 2 Calculate surface area of a cone with radius of 3cm and a height of 7cm.

Solution

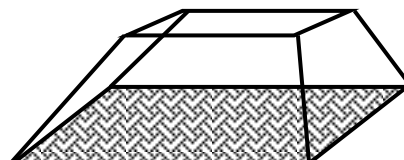
$$\begin{aligned} V &= \pi r^2 + \pi rh \\ &= \frac{22}{7} \times 3 \times 3 + \frac{22}{7} \times 3 \times 7 \\ &= \frac{198}{7} + \frac{462}{7} \\ &= \frac{660}{7} \text{ cm}^2 \\ &= 94 \frac{2}{7} \text{ cm}^2 \end{aligned}$$

Frustum

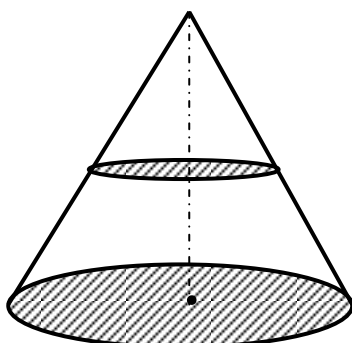
A part of a solid such as a cone or pyramid lying between a base face and a parallel face to base face that intersects with solid.



Frustum of a right- cone



Frustum of a rectangular pyramid



Volume of a cone is found by $V = \frac{1}{3} r^2 h$. Volume of a frustum is found by subtracting the volume of the top cone from the whole cone. Thus volume of frustum is calculated as $V = \frac{1}{3} R^2 H - \frac{1}{3} r^2 h$.

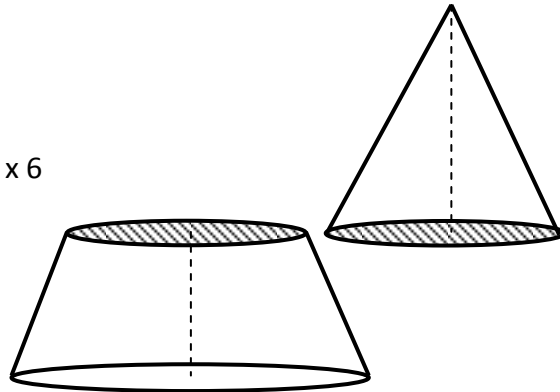
Note that R is radius and H is the height of base face and r is radius h is the height of top cone.



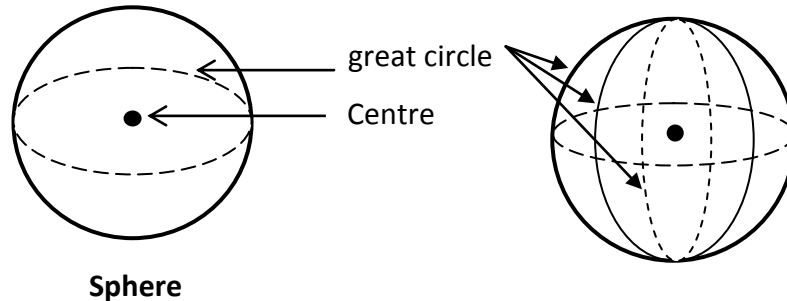
Example Calculate volume of the frustum if cone of radius 10 cm and a height of 15 cm is cut horizontally 9 cm. The radius formed in the top cone is 4 cm.

Solution

$$\begin{aligned}
 V &= \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 h \\
 &= \frac{1}{3} \times 3.14 \times 10^2 \times 15 - \frac{1}{3} \times 3.14 \times 4^2 \times 6 \\
 &= 3.14 \times 500 - 3.14 \times 32 \\
 &= 1570 - 100.48 \\
 &= 1469.52 \text{ cm}^2
 \end{aligned}$$

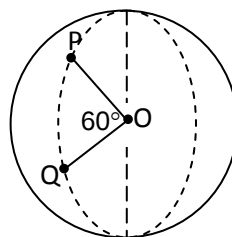


Just like a ball, a **sphere** is a space or solid figure whose points are equidistant from its centre. A **great circle** of a sphere is the intersection between the sphere and a plane passing through the centre of the sphere. If we cut the sphere into four congruent parts, each part is called a **quadrant**. The total surface area of a quadrant equals the area of a great circle of a sphere. Since there are four quadrants, then the total surface area of a sphere is four times the area of its great circle.



The longitudes and the equator are great circles. All other latitudes above and below the equator are small circles.

Distance between two Points on a Sphere



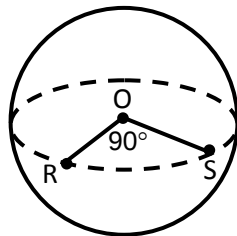
Example 1 The great circle contains P and Q in the arc PQ which subtends the angle of 65° at the centre or has an angular distance of 65° . The radius of the earth is about 6 400 km. Find the spherical distance.

**Solution**

The spherical distance is the arc length.

$$\begin{aligned}l &= \frac{\theta}{360^\circ} \times 2\pi r \\&= \frac{60}{360} \times 2 \times 3.14 \times 6400 \\&= 6698.67 \text{ km}\end{aligned}$$

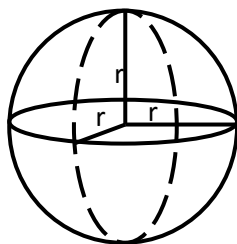
Example 2 The great circle (equator) contains R and S in the arc RS which subtends the angle of 90° at the centre. The radius of the earth is about 6 400 km. Find the spherical distance between RS.

**Solution**

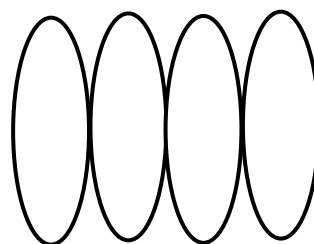
$$\begin{aligned}l &= \frac{\theta}{360^\circ} \times 2\pi r \\&= \frac{90^\circ}{360^\circ} \times 2 \times 3.14 \times 6400 \\&= \frac{1}{4} \times 2 \times 3.14 \times 6400 \\&= 3.14 \times 3200 \\&= 10048 \text{ km}\end{aligned}$$

Volume and Surface Area of Sphere

$$V = \frac{4}{3}\pi r^3$$



$$SA = 4\pi r^2$$





Since the equator and all longitudes are great circles, their distance to the centre are equal to the distance from the centre to the pole, thus are expressed as r .

The cube of the radii only gives a third the volume of a quarter sphere, therefore we multiply by four to obtain the volume of the sphere.

Example 1 Calculate volume and surface area of a sphere with radius of 3 m.

Solution

$$\begin{aligned} \text{a) } V &= \frac{4}{3} \pi r^3 \\ &= \frac{4}{3} \times 3.14 \times 3^3 \\ &= 4 \times 3.14 \times 3^2 \\ &= 113.04 \text{ m}^3 \end{aligned} \qquad \begin{aligned} \text{b) } SA &= 4\pi r^2 \\ &= 4 \times 3.14 \times 3^2 \\ &= 3.14 \times 36 \\ &= 113.04 \text{ m}^2 \end{aligned}$$

Example 2 Calculate spherical distance on the equator given that the angular distance is 30° .

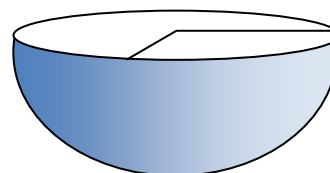
Solution

$$\begin{aligned} l &= \frac{\theta}{360} \cdot 2\pi r \\ &= \frac{30}{360} \times 2 \times 3.14 \times 6400 \\ &= \frac{1}{6} \times 3.14 \times 6400 \\ &= \frac{20096}{6} \\ &= 3349.33 \text{ km} \end{aligned}$$

Example 3 Calculate circular area covered by the great circle plane. Use $r = 6\,400\text{ km}$.

Solution

$$\begin{aligned} A &= \pi r^2 \\ &= 3.14 \times 6400 \times 6400 \\ &= 3.14 \times 40\,960\,000 \\ &= 128\,614\,400 \text{ km}^2 \end{aligned}$$



**LEARNING ACTIVITY 11.4.4.4**

20 minutes

A. Identification: Write the correct answer before each number.

- _____ 1. The amount of substance that a solid contain.
- _____ 2. A three dimensional figure that has two congruent polygons lying in parallel planes and bases and whole lateral faces are parallelograms.
- _____ 3. A three dimensional figure with six congruent lateral faces.
- _____ 4. What geometric figure is formed by the intersections of the edges of a solid figure?
- _____ 5. A three dimensional figure with a circular base and a vertex.
- _____ 6. If the sphere is to be divided into two hemispheres, which circle contains the diameter of the sphere?
- _____ 7. A space figure that has a polygon for its base and triangles for its lateral faces.
- _____ 8. The segment connecting the vertex and centre of the base of a cone or pyramid.
- _____ 9. A three dimensional figure with two congruent circular bases.
- _____ 10. A space or solid figure whose points are equidistant from its centre.

B. Problem Solving.

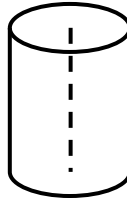
1. Find the lateral area, total area and volume of a rectangular prism whose length is 26 cm, width is 8 cm and whose height is 5 cm.



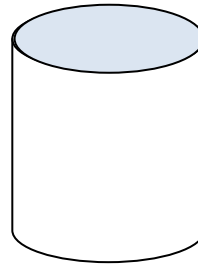
2. Find the lateral area, total area and volume of the given cylinder.

Given: Circumference = 8π

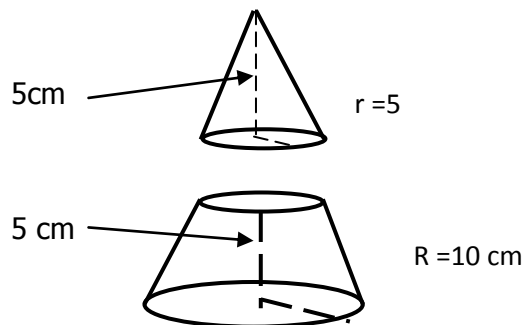
Height = 10 cm



3. Calculate volume of the content given that $\pi = \frac{22}{7}$, $h = 1.2$ m and $r = 3.5$ cm .
Give your final answer in Litres.



4. Find volume of the frustum using $\pi = 3.14$.



5. Calculate the spherical distance on the great circle if the angular distance is 120° .
Use $r = 6\,400$ km and $\pi = 3.14$

**SUMMATIVE TASKS 11.4.4**

20 minutes

A. Multiple Choice - Write the letter of your choice on the blank before each number.

_____ 1. A big bottle of contains 1.5L of soda. Which of the following is described by 1.5L?

- a. area b. circumference c. perimeter d. volume

_____ 2. Using a non-standard unit, which is more accurate to be used in getting the volume?

- a. marbles b. sand c. both d. Neither

_____ 3. What is the altitude of a lateral face of the pyramid?

- a. axis b. height c. lateral height d. slant height

_____ 4. Given a cube with edge 8 cm long, how long is the diagonal of each face?

- a. 4 cm b. 5.65 cm c. 5.56 cm d. 11.31 cm

_____ 5. An aquarium is 12.5 dm long, 4.5 dm wide and 65 cm high. How much water can it hold

if the water is 10cm less from the rim?

- a. 365.625 dm^3 b. 3565.25 dm^3 c. 3093.75 dm^3 d. 309.375 dm^3

_____ 6. The segment connecting the vertex and the center of the base of a cone or a pyramid.

- a. axis b. lateral edge c. height d. slant height

_____ 7. The amount of substance that a solid contain.

- a. area b. circumference c. perimeter d. volume

_____ 8. A space figure that has a polygon for its base and whose lateral faces are triangles.

- a. hemisphere b. prism c. pyramid d. Sphere

_____ 9. A solid figure bounded by planes.

- a. polygon b. polyhedron c. hemisphere d. atmosphere

_____ 10. What are the intersections of the edges of a solid figure?

- a. vertices b. lines c. planes d. space

_____ 11. . Which of the following is a circular cylinder whose elements are perpendicular to its base?



- a. Right circular cylinder
- b. Congruent cylinder
- c. Oblique cylinder
- d. Oblique circular cylinder

_____ 12. Which of the following solid figures has a polygonal base of any number of sides and the other faces are triangles which have a common vertex?

- a. prism
- b. pyramid
- c. cone
- d. polyhedron

_____ 13. If the base of a cylinder is congruent to the base of a cone and whose altitude are equal, then the volume of the cone is _____ the volume of the cylinder.

- a. equal
- b. one-half
- c. one-third
- d. one-fourth

_____ 14. What geometric figure is formed by the lateral faces of a prism?

- a. parallelogram
- b. rectangle
- c. square
- d. triangle

_____ 15. If the base of a pyramid is a regular polygon, then what geometric figure are formed by the lateral faces of the pyramid?

- a. equilateral triangles
- b. equiangular triangles
- c. isosceles triangles
- d. scalene triangles

B. Problem Solving

1. The volume of a square pyramid is 936 cm^3 . The length of the altitude is 6.5 cm. Find the length of the side of the base.

2. A drinking cup shaped like a cone has a radius of 8.2cm and a height of 10.4cm. What is its volume?



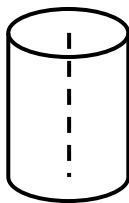
3. A rectangular prism has a length of 12in and width of 8in for its base. If the altitude of the prism is 10 in, find the volume, lateral area and total surface area.

4. Find the lateral area, total area and volume of a rectangular prism whose length is 22 cm, width is 5 cm and whose height is 7 cm.

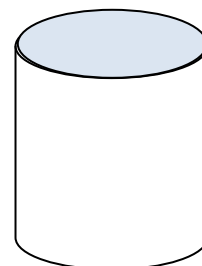
5. Find the lateral area, total area and volume of the given cylinder.

Given: Circumference = 10π

Height = 8 cm

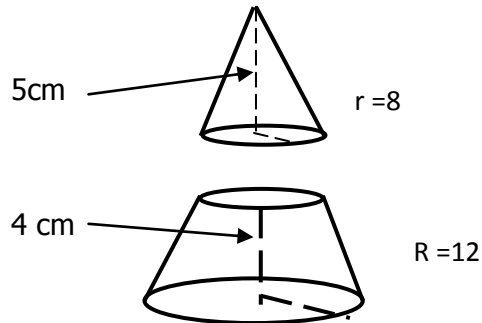


6. Calculate volume of the content given that $\pi = \frac{22}{7}$, $h = 0.8$ m and $r = 14$ cm .
Give your final answer in Litres.



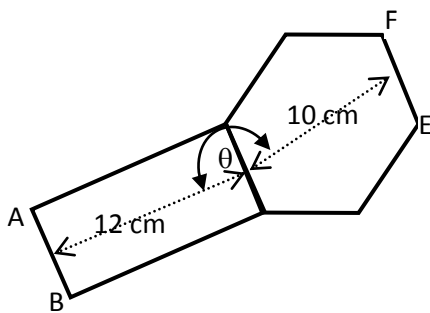


7. Find volume of the frustum ($\pi = 3.14$).



8. Calculate the spherical distance on the great circle if the angular distance is 270° .
Use $r = 6\,400\text{ km}$ and $\pi = 3.14$

9. Calculate dihedral angle marked θ below, given that horizontal distance between AB and EF is 6 cm . Use Cosine Rule.

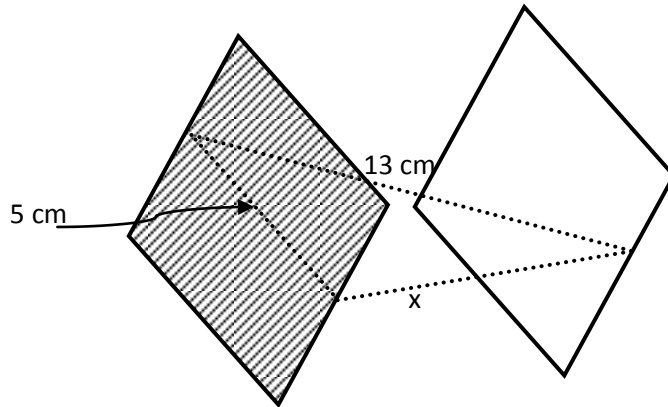


10. Draw a plane parallel to plane given.

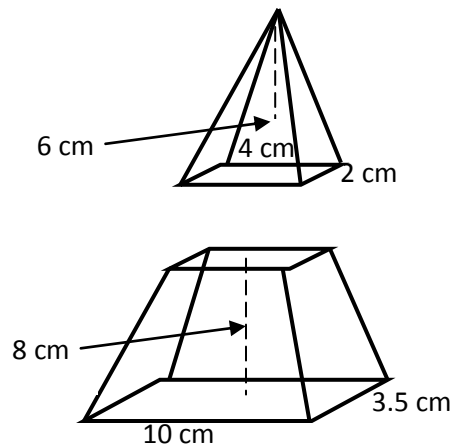




11. Use Pythagoras Theorem to calculate the horizontal distance x between the two parallel planes. $c^2 = a^2 + b^2$



12. Find volume of the frustum.





SUMMARY

This summary outlines the key ideas and concepts to be remembered

- **Geometry** is the study of shapes and figures as well as their properties. It comes from the Greek words *geo* which means “earth” and *metron* which means “to measure”.
- Three undefined terms basic to Geometry are point, line and plane.
 - A **point** represents an exact location in space. It has no dimensions.
 - A **line** is a set of continuous points that extends indefinitely in either direction.
 - A **plane** is represented by a flat surface that extends indefinitely.
- An **angle** is a geometric figure formed by two rays that have a common endpoint. Kinds of angles are acute, obtuse and right angles. Angle pairs are classified as complementary, supplementary, vertical, adjacent and linear pairs.
- A **polygon** is a simple closed plane figure made up of line segments. Types of polygons can be identified according to the number of sides.
- A **triangle** is a polygon with three sides.
 - Triangles according to angle measures are:
 - ◆ **Right triangle** has one right angle.
 - ◆ **Obtuse triangle** has one obtuse angle.
 - ◆ **Acute triangle** has three acute angles.
 - ◆ **Equiangular triangle** has three equal angles.
 - Triangles according to side measures are:
 - ◆ **Scalene triangle** has no equal side measure.
 - ◆ **Isosceles triangle** has at least two equal sides.
 - ◆ **Equilateral triangle** has three equal sides.
- A **quadrilateral** is a polygon with four sides, four angles and four vertices.
 - ◆ A quadrilateral with both pairs of opposite sides parallel is called a **parallelogram**.
 - ◆ A quadrilateral with four right angles is called a **rectangle**.
 - ◆ A quadrilateral having four equal sides is called a **rhombus**.
 - ◆ A quadrilateral having four equal sides and four right angles is a **square**.
 - ◆ A quadrilateral with exactly one pair of opposite sides parallel is a **trapezoid**.
- The sum of the interior angles of a polygon = $(n - 2) \cdot 180^\circ$ where n = the number of sides.



- **The sum of the exterior angles of any polygon is 360° .**
- A **proof** is a sequence of true statements placed in logical order. To show the validity of the proof, a reason that justifies each statement must be provided.
- **CPCTC (Corresponding Parts of Congruent Triangles are Congruent)**
If two triangles are congruent, then the six pairs of corresponding parts are congruent.
- **Corresponding angles** are pairs of angles whose vertices are paired in a given correspondence between two triangles.
- **Corresponding sides** are pairs of segments whose endpoints are vertices that are paired in a given correspondence between two triangles. Corresponding sides of triangles lie opposite corresponding angles.
- **Postulates** are statements that are accepted as true without proof.
- **Theorems** are statements that need to be proven.
- **Side-Side-Side or SSS Postulate.** If three sides of one triangle are equal to the corresponding sides of another triangle, then the two triangles are congruent.
- **Side-Angle-Side or SAS Postulate.** If two sides and the included angle of one triangle are congruent to the corresponding parts of another triangle, then the two triangles are congruent.
- **Angle-Side-Angle or ASA Postulate.** If two sides and the included angle of one triangle are congruent to the corresponding parts of another triangle, then the two triangles are congruent.
- **Angle-Angle-Side (AAS) or Side-Angle-Angle (SAA) Theorem.** If two consecutive angles and a side opposite one of the given angles are congruent to two angles and the corresponding side opposite one of the angles, then the two triangles are congruent. AAS or SAA is a theorem because it can be proven using the other triangle congruence postulates.
- **Isosceles Triangle Theorem 1.** If two sides of a triangle are congruent, then the angles opposite those sides (base angles) are congruent.
- **Isosceles Triangle Theorem 2.** If two angles of a triangle are congruent, then the sides opposite the congruent angles are congruent.
- **Equilateral Triangle Theorem.** If a triangle is equilateral, then it is also equiangular.
- **Equiangular Triangle Theorem.** If a triangle is equiangular, then it is also equilateral.



- **Leg-Leg (LL) Theorem.** If the legs of one right triangle are congruent to the legs of another right triangle, then the two triangles are congruent.
- **Leg-Angle (LA) Theorem.** If a leg and an acute angle of one right triangle are congruent to the leg and an acute angle of another right triangle, then the two triangles are congruent.
- **Hypotenuse-Angle (HA) Theorem.** If the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and an acute angle of another right triangle, then the two triangles are congruent.
- **Hypotenuse-Leg (HL) Theorem.** If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and corresponding leg of another right triangle, then the two triangles are congruent.
- **Similar Triangles.** Two triangles are similar if and only if their vertices can be made to correspond in such a way that corresponding angles are congruent and the corresponding sides are proportional.
- **AA Postulate.** If two angles of a triangle are congruent to two angles of another triangle, then the two triangles are similar.
- **AAA Similarity Theorem.** If the corresponding angles of two triangles are congruent, then the two triangles are similar.
- **SSS Similarity Theorem.** If the lengths of the three sides of one triangle are proportional to the corresponding three sides of another triangle, then the two triangles are similar.
- **SAS Similarity Theorem.** If the lengths of the two pairs of corresponding sides of two triangles are proportional and their respective included angles are congruent, then the two triangles are similar.
- A **circle** is the set of all points equidistant from a fixed point called the **centre** while the fixed distance is the **radius**. Parts of a circle are as follows:
 - A **radius** is a segment drawn from any point of the circle to its centre.
 - A **chord** is a segment connecting any two points on the circle.
 - A **diameter** is a chord passing through the centre.
 - A **secant** is a line, ray or segment that contains a chord. It usually passes through two points on the circle.
 - A **tangent** is a line, ray or a segment passing through exactly one point on the circle. The point of contact is called the **point of tangency**.
 - A **central angle** is an angle whose vertex is the centre of the circle
 - The **measure of a central angle** is proportional to the measure of the arc it intercepts. The **measure of a minor arc** is equal to the measure of its central angle. The **measure of a major arc** is equal to 360° - measure of the minor arc.



- The **measure of a semi circle** is 180° . Note that two circles congruent of their radii are equal or congruent.
- **Arc Addition Postulate.** The measure of the arc formed by two adjacent arcs is the sum of the measures of the two arcs. Adjacent arcs of a circle are arcs that share a common endpoint.
- **Central Angle-Arc Theorem.** In the same circle or in congruent circles, two arcs are congruent if their central angles are congruent.
- **Arc-Central Angle Theorem.** In the same circle or in congruent circles, two central angles are congruent if their corresponding minor arcs are congruent.
- An **inscribed angle** is an angle whose vertex is on the circle and whose sides contain chords of the circle.
- **Inscribed Angle Theorem.** An inscribed angle is equal in measure to one half its intercepted arc.
- **Angle diameter theorem.** If an inscribed angle intercepts a semicircle, then the angle is a right angle
- **Congruent Arc Theorem.** In the same or congruent circles, inscribed angles with congruent intercepted arcs are congruent.
- **Congruent Angle Theorem.** In the same or congruent circles, congruent inscribed angles have congruent intercepted arcs.
- **Arc-Chord Theorem .** In the same circle or congruent circles, congruent arcs have congruent chords.
- **Radius Bisects Chord Theorem.** In a circle, if a radius bisects a chord and its arc, then the radius is perpendicular to the chord.
- **Diameter Perpendicular Bisector to Chord Theorem.** If a diameter of a circle is perpendicular to a chord, then the diameter bisects the chord and the arc.
- **Chord Bisects Arc Theorem.** If a chord of a circle bisects a second chord and its arc, then the first chord is a diameter and is perpendicular to the second chord.
- A quadrilateral is said to be a **cyclic quadrilateral** if there is a circle passing through all of its four vertices.



- **Tangent Line Theorem.** If a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of tangency.
- **Secant-Secant Theorem.** If two secants intersect in the interior of a circle, then the measure of an angle formed is equal to one-half the sum of the measures of the intercepted arcs. This theorem is also true for chords.
- **Secant-Tangent Theorem (Case 1).** When a tangent and a secant (or chord) intersect in a point on a circle, the measure of the angle formed is one-half the degree measure of the intercepted arc.
- **Secant-Tangent Theorem (Case 2).** When two secants, a tangent and a secant or two tangents intersect in the exterior of a circle, the measure of the angle formed is one-half the difference of the measures of the intercepted arc.
- A three-dimensional object is a solid figure bounded by intersecting planes called **polyhedron**.
 - A **prism** is a polyhedron with two opposite faces that are congruent and parallel. The other faces are parallelogram
 - A **pyramid** is a polyhedron whose lateral faces have one common vertex.
 - A **cylinder** is a space figure whose bases are congruent and are parallel circles.
 - a **sphere** is a space or solid figure whose points are equidistant from its centre. A **great circle** of a sphere is the intersection between the sphere and a plane passing through the center of the sphere.
- **Dihedron** is shape formed by two planes.
- **Dihedral angle** is the angle formed by two planes, whose arms are perpendicular to respective planes.



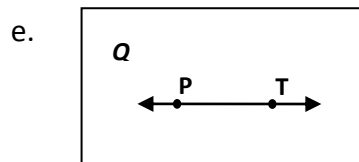
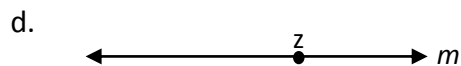
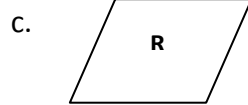
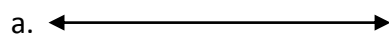
ANSWERS TO LEARNING ACTIVITIES

STUDENT LEARNING ACTIVITY 11.4.1.1 (p. 9-10)

1.

a. point	f. line
b. line	g. line
c. line	h. point
d. plane	i. point
e. plane	j. plane
2.
 - a. true
 - b. true
 - c. false
 - d. false
 - e. false

3.

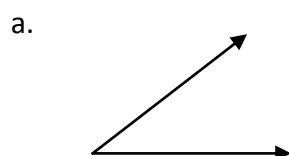
**STUDENT LEARNING ACTIVITY 11.4.1.2.1 (p. 17-18)**

1.
 - a. acute angle
 - b. obtuse angle
 - c. right angle
 - d. acute angle
 - e. obtuse angle

2.

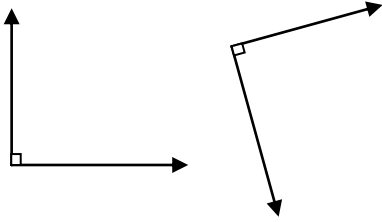
- a. 90°
- b. 125°
- c. 60°
- d. 100°
- e. 240°

3. Note: answers may vary

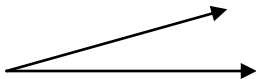




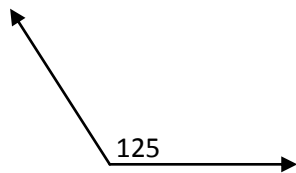
b.



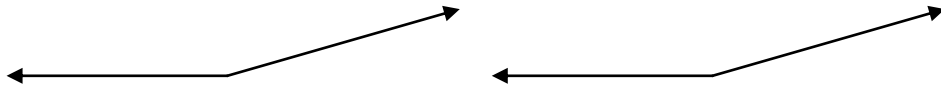
c.



d.



e.



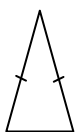
4. a. False
b. False
c. False
d. False
e. True

STUDENT LEARNING ACTIVITY 11.4.1.2.2 b (p. 23-24)

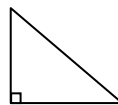
1. a. $\triangle BCE$ f. $\triangle CDE$ (or $\triangle BCE$)
b. $\triangle DEA$ (or $\triangle CEA$) g. $\overline{BD}$ (or $\overline{DB}$)
c. No h. $\triangle BDE$
d. Yes i. 6 (six)
e. Yes j. $\triangle ADE$, $\triangle BDE$, $\triangle ACE$, $\triangle DCE$, $\triangle BCE$, $\triangle ABE$

2. Note: Answers may vary

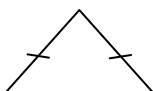
a.



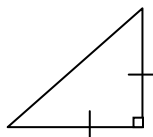
b.

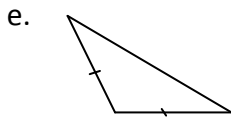


c.



d.





STUDENT LEARNING ACTIVITY 11.4.1.3 (p. 28)

1. 79° (exterior angles are 133° , 126° and 101°)
2. 50° , 50° , 80°
3. 30° , 60°
4. 120°
5. 120°
6. 107°
7. 15°

STUDENT LEARNING ACTIVITY 11.4.1.4 (p.38)

1. a. True f. False
b. True g. False
c. False h. False
d. true i. True
e. True j. True
2. 18
3. (i) opposite sides are parallel and equal (ii) opposite angles are congruent
4. A rectangle is a parallelogram because it has all properties of parallelogram.
5. Rhombus

STUDENT LEARNING ACTIVITY 11.4.1.5 (p. 42)

1. a. 1620° 6. $x = 54^\circ$ $y = 36^\circ$
b. 1800°
2. a. 135° 7. $x = 60^\circ$ $y = 60^\circ$ $z = 120^\circ$
b. 144°
3. a. 120° 8. $(18 + 18\sqrt{2}) \text{ cm}^2$
b. 30°
4. 111° 9. $x = 90$
5. 3240° 10. 19.1 cm^2

SUMMATIVE TASK 11.4.1 Summary (p. 49-50)

A. Multiple Choice

- | | | | |
|------|-------|-------|-------|
| 1. b | 6. b | 11. c | 16. d |
| 2. b | 7. c | 12. b | 17. c |
| 3. d | 8. d | 13. b | 18. d |
| 4. b | 9. b | 14. b | 19. b |
| 5. c | 10. c | 15. b | 20. b |

B. Problem Solving

1. 32 cm
2. 24.4 cm^2
3. 132°



4. (i) sum of interior $180.(n-2)$ (ii) interior of regular $S \div 10$ (iii) $180 - \text{interior} = \text{exterior}$
5. 1260°
6. 16 sides or hexadecagon
7. $X = 120^\circ$ $y = 90^\circ$ $z = 60^\circ$
8. 228°

STUDENT LEARNING ACTIVITY 11.4.2.1 to 11.4.2.2 (p. 62 - 63)

1. a. Statement: $\triangle ABD \cong \triangle CDB$
Reason : SSS Postulate
- b. Statement: $\triangle EFG \cong \triangle IHG$
Reason : SAS Postulate
- c. Statement: $\triangle JLM \cong \triangle JLK$
Reason : SAS Postulate
- d. Statement: $\triangle NOP \cong \triangle PQN$
Reason : ASA Postulate
- e. Statement: $\triangle RSV \cong \triangle TSU$
Reason : SAS Postulate

2. Prove: $\overline{WX} \cong \overline{YX}$

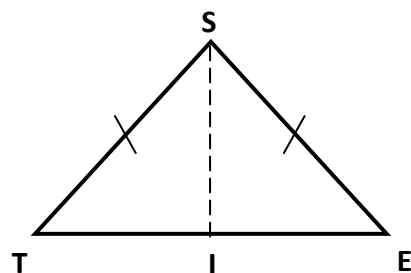
Proof:

Statements	Reasons
1) $\angle W$ and $\angle Y$ are right angles.	1) Given
2) $\overline{ZX}$ bisects $\angle WXY$.	2) Given
3) $\angle WXZ \cong \angle YXZ$	3) Definition of angle bisector
4) $\overline{ZX} \cong \overline{ZX}$	4) Reflexive Property
5) $\triangle WXZ \cong \triangle YXZ$	5) AAS Theorem
6) $\overline{WX} \cong \overline{YX}$	6) CPCTC

3. Write a two-way proof.

Given: $\triangle SET$ is an isosceles triangle with
 $\overline{ST} \cong \overline{SE}$
 $\overline{SI}$ bisects $\angle S$.

Prove: $\angle T \cong \angle E$



Statements	Reasons
1) $\triangle SET$ is an isosceles triangle with $\overline{ST} \cong \overline{SE}$	1) Given
2) $\overline{SI}$ bisects $\angle S$.	2) Given
3) $\angle TSI \cong \angle ESI$	3) Definition of angle bisector
4) $\overline{SI} \cong \overline{SI}$	4) Reflexive Property
5) $\triangle TSI \cong \triangle ESI$	5) SAS Theorem
6) $\angle T \cong \angle E$	6) Base angles of an isosceles triangle are $\cong$ or CPCTC



STUDENT LEARNING ACTIVITY 11.4.2.3 (p. 66-67)

1.

Statements	Reasons
1) $\overline{AB} \cong \overline{BC}$ and $\overline{CD} \cong \overline{AD}$	1) Given
2) Draw diagonal $\overline{BD}$ or $\overline{AC}$	2) Two points determine a line
3) $\overline{BD} \cong \overline{AC}$	3) The diagonals of a square are $\cong$
4) $\overline{BD} \cong \overline{BD}$ and $\overline{AC} \cong \overline{AC}$	4) Reflexive Property
5) $\overline{CD} \cong \overline{AD}$	5) Given
6) $\triangle ABD \cong \triangle BCD$	6) SSS Congruence

2.

Statements	Reasons
1) $\angle P \cong \angle E$	1) Given
2) A is the midpoint of $\overline{PE}$	2) Given
3) $\overline{PA} \cong \overline{AE}$	3) Definition of midpoint
4) $\angle PAL \cong \angle EAC$	4) Vertical angles are congruent
5) $\triangle LPA \cong \triangle ECA$	5) ASA Postulate

STUDENT LEARNING ACTIVITY 11.4.2.4 (p. 70)

- | | |
|--------------------|---------------------|
| 1. $\overline{EC}$ | 6. $\overline{GC}$ |
| 2. $\overline{RA}$ | 7. $\overline{GA}$ |
| 3. $\overline{GA}$ | 8. $\overline{GR}$ |
| 4. $\overline{GC}$ | 9. $\overline{AC}$ |
| 5. $\overline{EG}$ | 10. $\overline{RG}$ |

STUDENT LEARNING ACTIVITY 11.4.2.5 (p. 74-75)

1. $\overline{AG}$
2. $\overline{TA}$
3. Yes, because it forms a balance equation.
4.
 - a. ✓
 - b. ✓
 - c. X
 - d. X

STUDENT LEARNING ACTIVITY 11.4.2.6 (p. 77-78)

1. 13 ft
2. 28 ft.
3. 1.05 ha



4. $330\,000\text{ m}^2$
5. 6 cm by 4 cm on paper
6. 79 m^2

SUMMATIVE TASK 11.4.2 (p. 79 – 81)**A. Multiple Choice**

- | | |
|------|-------|
| 1. d | 6. d |
| 2. a | 7. c |
| 3. b | 8. c |
| 4. c | 9. b |
| 5. a | 10. b |

B. Problem Solving

1. $x = 24$ $y = 19.5$ (or $39/2$)
2. $\frac{1}{2}$
3. $AB = 5\text{ cm}$
4. $x = 15\text{ cm}$
5. 62.5 m^2

STUDENT LEARNING ACTIVITY 11.4.3.1 to 11.4.3.2 (p. 90-92)**A. True or False**

- | | |
|----------|-----------|
| 1. False | 11. True |
| 2. True | 12. False |
| 3. False | 13. True |
| 4. True | 14. True |
| 5. True | 15. False |
| 6. True | 16. True |
| 7. True | 17. True |
| 8. False | 18. True |
| 9. True | 19. False |
| 10. True | 20. True |

- B.
1. $40^\circ, 43^\circ, 90^\circ$
 2. 18 cm
 3. 60°
 4. $\frac{4000}{\pi}\text{ cm}^2$
 5. 60°

STUDENT LEARNING ACTIVITY 11.4.3.3 (p. 75-76)

- A.
1. Yes
 2. No
 3. Yes
 4. No



5. No

6. No

- B.
1. 114°
 2. 75°
 3. $15\frac{1}{3}$ cm
 4. $\sqrt{11}$ cm or 3.3 cm
 5. 2.1 cm
 6. 0.86 cm
 7. $\frac{16\sqrt{3}}{3}$ cm or 9.2 cm
 8. CD = 24 cm
 9. x = 40 cm

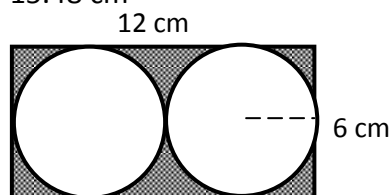
STUDENT LEARNING ACTIVITY 11.4.3.4 to 11.4.3.5 (p. 107-111)

A. Multiple Choice

- | | |
|------|------|
| 1. b | 6.b |
| 2. b | 7.b |
| 3. c | 8.c |
| 4. b | 9.b |
| 5. c | 10.b |

Part B

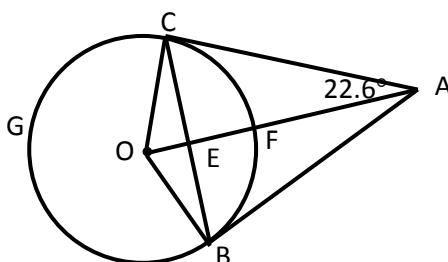
1. 22 cm^2
2. 117°
3. 2.4 cm
4. $y = \sqrt{7}$ cm
5. a. 11 cm b. 5 cm c. $4\sqrt{6}$ d. $11 + 4\sqrt{6}$
6. $z = 2\sqrt{3}$
7. Shaded Area = 15.48 cm^2



8. $x = 40^\circ$
9.
 - a. Chord LM = 6 cm
 - b. Line Segment AN = 6 cm
 - c. Altitude CL = $12\sqrt{3}$ or 10.4
 - d. Angle LMN = 60°
 - e. Angle NOM = 60°



- 10 a. $\angle ACO = 90^\circ$
b. $\angle AEC = 90^\circ$
c. $\angle COA = 67.4^\circ$
d. $\angle OBE = 22.6^\circ$
e. $\angle EBA = 67.4^\circ$
f. $\angle BOC = 134.8^\circ$
g. Reflex $\angle BOC = 225^\circ$
h. $\angle BGC = 67.4^\circ$
i. $OA = 13$ cm
j. $OF = 2.5$ cm
k. $OE = 2.4$ cm
l. $BC = 9.2$ cm
m. $CE = 4.6$ cm
n. Arc length $BF = 5.9$ cm

**STUDENT LEARNING ACTIVITY 11.4.3.6 (p. 115)**

- | | |
|----------------|----------------|
| a. 90° | f. 125° |
| b. 115° | g. 90° |
| c. 113° | h. 70° |
| d. 80° | |
| e. 85° | |

SUMMATIVE TASK 11.4.3 (p. 116-118)**A. Identification**

- | | |
|----------------------|-----------------|
| 1. diameter | 6. outside/on |
| 2. chord | 7. 90° |
| 3. point of tangency | 8. radius |
| 4. $x^2 + y^2 = 1$ | 9. arc |
| 5. 2 units | 10. Semi-circle |

B. Problem Solving

- a. $x = 33^\circ$
b. $x = 41^\circ$
- $\angle UYW = 36^\circ$
 $\angle UYX = 80^\circ$
 $\angle WYX = 44^\circ$
- $\sqrt{2}$
- 6 cm
- 12 cm^2
- $\angle ABC = 130^\circ$
- $\angle OAD = 60^\circ$
- 43.3 cm^2

STUDENT LEARNING ACTIVITY 11.4.4.1 (p. 120)



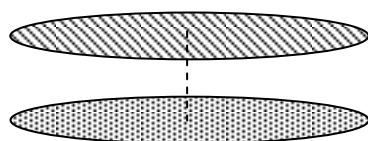
- | | |
|----------|----------|
| 1. True | 6. False |
| 2. True | 7. True |
| 3. False | 8. False |
| 4. True | 9. True |
| 5. False | 10. True |

STUDENT LEARNING ACTIVITY 11.4.4.2-11.4.4.3 (p. 124-125)

- | | |
|----------|----------|
| 1. True | 6. True |
| 2. False | 7. True |
| 3. False | 8. True |
| 4. True | 9. False |
| 5. True | 10. True |

Part B.

1. 150.72 cm^3
2. 34.24 cm^3
3. 34.9° or 35°
4. ..



5. 8cm

STUDENT LEARNING ACTIVITY 11.4.4.4 (p. 136-137)

A. Identification

- | | |
|-----------|----------------|
| 1. volume | 6. semi-circle |
| 2. prism | 7. pyramid |
| 3. cube | 8. axis |
| 4. lines | 9. cylinder |
| 5. cone | 10. sphere |

B. Problem Solving

1. Lateral Area = 832 cm^2
Total Surface Area = 912 cm^2
Volume = 1040 cm^3
2. Lateral Area = $80\pi \text{ cm}^2$
Total Surface Area = $96\pi \text{ cm}^2$
Volume = $160\pi \text{ cm}^3$
3. 4.62L
4. 2747.5 cm^3
5. $13\,397.3 \text{ km}$ or $13\,397\frac{1}{3} \text{ km}$

SUMMATIVE TASK 11.4.4 (pages 138-142)



A . Multiple Choice

- | | |
|-------|-------|
| 1. d | 11. a |
| 2. b | 12. b |
| 3. d | 13. c |
| 4. d | 14. a |
| 5. c | 15. c |
| 6. a | |
| 7. d | |
| 8. c | |
| 9. b | |
| 10. b | |

B. Problem Solving

- side = 12 cm
- Lateral Area = 400 cm^2
Total Surface Area = 496 cm^2
Volume = 960 cm^3
- $V = 960 \text{ in}^3$, $SA = 592 \text{ cm}^2$
-
- $80\pi \text{ cm}^2$ or 251.2 cm^2
- 49.28L
- 1021.55 cm^3
- 30 144 km
- 41°
-



- 12 cm
- 159 cm^3

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